Quantum Information Theory, Spring 2019

Exercise Set 9 **in-class practice problems**

1. **Warmup:**

   (a) Show that every classical state $\rho_{\mathsf{XY}} \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ is separable.

   (b) Show that a pure state $|\psi\rangle_{\mathsf{XY}} \in \mathcal{S}(\mathcal{X} \otimes \mathcal{Y})$ is separable if and only if its entanglement entropy is zero.

   (c) In the lecture we defined *separable operators* to be those of the form

   $$\sum_i P^i_{\mathsf{X}} \otimes Q^i_{\mathsf{Y}}$$

   for $P^i_{\mathsf{X}}, Q^i_{\mathsf{Y}}$ positive. Show that the restriction of this definition to density matrices coincides with the definition of separable states from the lecture.

   (d) Recall that the four *Bell states* are defined by

   $$|\Phi^{zx}\rangle = (Z^z X^x \otimes I)|\Phi^+\rangle.$$

   where $|\Phi^+\rangle$ is the canonical two-qubit maximally entangled state. Verify that

   $$|\Phi^{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |\Phi^{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
   $$|\Phi^{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \qquad |\Phi^{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

   Verify also that

   $$|\Phi^{zx}\rangle = (I \otimes X^x Z^z)|\Phi^+\rangle.$$

2. **Two-qubit pure states (product vs entangled):** Let

   $$|\psi\rangle = \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix} \in \mathbb{C}^4.$$

   be an arbitrary pure state on two qubits. Define

   $$\Delta(|\psi\rangle) = \psi_{00}\psi_{11} - \psi_{01}\psi_{10}.$$

   The goal of this exercise is to show that $\Delta(|\psi\rangle) = 0$ if and only if $|\psi\rangle$ is a product state.

   (a) Assume that $|\psi\rangle$ is a product state, i.e., $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$, for some single-qubit states

   $$|\alpha\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}, \qquad |\beta\rangle = \begin{pmatrix} \Phi^0 \\ \Phi^1 \end{pmatrix}.$$

   Show that in such case $\Delta(|\psi\rangle) = 0$.

(b) Conversely, let $|\psi\rangle$ be an arbitrary two-qubit state and assume that $\Delta(|\psi\rangle) = 0$. Find two single-qubit states $|\alpha\rangle$ and $|\beta\rangle$ such that $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$.

3. **Pauli matrices and the swap:** Let $\Sigma = \{0, 1\}$ and $\mathcal{X} = \mathcal{Y} = \mathbb{C}^\Sigma$. The two-qubit *swap operation* $W \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Y})$ is defined on the computational basis states as follows:

$$W|a, b\rangle = |b, a\rangle,$$

for all $a, b \in \{0, 1\}$. Recall that the four *Pauli matrices* are

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Verify that

$$W = \frac{1}{2}(I \otimes I + X \otimes X + Y \otimes Y + Z \otimes Z).$$

(b) Verify that

$$W = \frac{1}{2} \sum_{z,x \in \{0,1\}} Z^z X^x \otimes X^x Z^z.$$

4. **Superdense coding:** Assume that Alice and Bob each have one qubit and their joint state is $|\Phi^{00}\rangle$. Assume that Alice wants to send two bits $z, x \in \{0, 1\}$ to Bob. They perform the following protocol: (i) Alice applies some unitary operation on her qubit, (ii) sends her qubit to Bob, and (iii) Bob performs an orthogonal measurement to recover z and x.

(a) What operation should Alice apply?
(b) What measurement should Bob perform?
(c) Formulate this procedure as a resource trade-off.