Quantum Information Theory, Spring 2019

Exercise Set 5 **in-class practice problems**

1. **Monotonicity of the fidelity and trace distance:** On Problem Set 2, you proved that the fidelity function has the following monotonicity property: If $\rho, \sigma \in D(\mathcal{X} \otimes \mathcal{Y})$ then

$$F(\rho_X, \sigma_X) \geq F(\rho, \sigma),$$

where $\rho_X = \operatorname{Tr}_Y[\rho]$, $\sigma_X = \operatorname{Tr}_Y[\sigma]$.

   (a) Prove that $\|\operatorname{Tr}_Y[A]\|_1 \leq \|A\|_1$ for all $A \in L(\mathcal{X} \otimes \mathcal{Y})$.

   Hint: Use the formula $\|A\|_1 = \max_{\|B\|_\infty \leq 1} |\operatorname{Tr}[AB]|$ *from Lecture 2.*

   (b) Conclude that the trace distance enjoys a similar monotonicity property (cf. Lecture 2):

$$\frac{1}{2}\|\rho_X - \sigma_X\|_1 \leq \frac{1}{2}\|\rho - \sigma\|_1$$

   (c) Why does the latter inequality go the other way around?

2. **Fidelity and trace distance:** For any ρ, $\sigma \in D(\mathcal{X})$, prove that

$$F(\rho, \sigma) \leq \sqrt{1 - \frac{1}{4}\|\rho - \sigma\|_1^2}.$$

Hint: Combine Uhlmann's theorem, Homework Problem 1, and the monotonicity property of the trace distance.

3. **Lossy vs. lossless compression:** In class, we mostly discussed *lossy (or fixed-length) compression protocols*, which compress sequences emitted by a source into bitstrings of fixed length but may fail with some small probability. In practice, it is also interesting to consider *lossless (or variable-length) compression protocols*, which succeed always but may emit strings of any length.

Given an (n, R, δ)-code as defined in class (which achieves lossy compression into bitstrings of length $\lfloor nR \rfloor$ with probability of failure $\leq \delta$), can you construct a lossless compression protocol with average rate close to R (for large n and small δ)?

4. **Lexicographic order:** The lexicographic order (denoted $\leq_{\text{lex}}$) on bitstrings of fixed length n is defined as follows: Given bitstrings x and y, we have that $x \leq_{\text{lex}} y$ if either $x = y$ or $x_i < y_i$ for the smallest i such that $x_i \neq y_i$. For example, `001` $\leq_{\text{lex}}$ `010`. The lexicographic order defines a total order on $\{0,1\}^n$, hence also on the set of bitstrings of length n with k ones, which we denote by $B(n, k)$.

   (a) Write down $B(5, 2)$ in lexicographically increasing order.
   (b) How can you recursively compute the m-th element of $B(n, k)$?
   (c) How can you recursively compute the index of a given element in $B(n, k)$?

Hint: $|B(n,k)| = \binom{n}{k}$. *Moreover,* $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ *for all* $1 \leq k \leq n-1$.