# Quantum Information Theory, Spring 2019

**Exercise Set 2** **in-class practice problems**

Throughout, $X$, $Y$, $Z$ denote quantum systems with Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$. The sets $\{|x\rangle\}$ and $\{|y\rangle\}$ denote arbitrary orthonormal bases of $\mathcal{X}$ and $\mathcal{Y}$, respectively.

1. **Computing reduced states:** Compute $\rho_X = \mathrm{tr}_Y[\rho]$ and $\rho_Y = \mathrm{tr}_X[\rho]$ in the following situations:

   (a) When $\rho = |\Psi\rangle\langle\Psi|$ is the two-qubit pure state given by

   $$|\Psi\rangle = \frac{1}{3}\big(|0,0\rangle + 2\,|0,1\rangle + 2\,|1,0\rangle\big) \in \mathcal{X} \otimes \mathcal{Y}$$

   and $\mathcal{X} = \mathcal{Y} = \mathbb{C}^2$. If this calculation seems too painful to carry out, use (c) below.

   (b) A classical state $\rho = \sum_{x,y} p(x,y)\,|x,y\rangle\langle x,y|$ corresponding to an arbitrary joint probability distribution $p(x,y)$.

   Now consider a general pure state $\rho = |\Psi\rangle\langle\Psi|$ given in the form $|\Psi\rangle = \sum_{x,y} A_{x,y}\,|x\rangle \otimes |y\rangle \in \mathcal{X} \otimes \mathcal{Y}$.

   (c) Verify that $\rho_X = AA^*$ and $\rho_Y = A^T\bar{A}$.

2. **Partial trace trickery:** Verify the following calculational rules for the partial trace. For all $A, B \in L(\mathcal{X})$, $M \in L(\mathcal{X} \otimes \mathcal{Y})$, and $N \in L(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z})$:

   (a) $\mathrm{tr}_Y[(A \otimes I_Y)M(B \otimes I_Y)] = A\,\mathrm{tr}_Y[M]B$,

   (b) $\mathrm{tr}_{XY}[N] = \mathrm{tr}_X[\mathrm{tr}_Y[N]] = \mathrm{tr}_Y[\mathrm{tr}_X[N]]$,

   What does the last rule look like if there is no $Z$-system?

3. **Schmidt decomposition:** In class, we discussed that any pure state $|\Psi\rangle = \sum_{x,y} A_{x,y}\,|x\rangle \otimes |y\rangle$ has a Schmidt decomposition.

   (a) Show that, if $\sum_i s_i\,|e_i\rangle\langle f_i|$ is a singular value decomposition of $A = \sum_{x,y} A_{x,y}\,|x\rangle\langle y|$, then $\sum_i s_i\,|e_i\rangle \otimes |\overline{f_i}\rangle$ is a Schmidt decomposition of $|\Psi\rangle$.

   (b) Find a Schmidt decomposition of the following two-qubit pure state:

   $$|\Psi\rangle = \frac{\sqrt{2}+1}{\sqrt{12}}(|00\rangle + |11\rangle) + \frac{\sqrt{2}-1}{\sqrt{12}}(|01\rangle + |10\rangle)$$

4. **Projective measurements:** Let $\mu\colon \Omega \to \mathrm{Pos}(\mathcal{X})$ be a *projective* measurement. Show that the projections $\mu(\omega)$ are pairwise orthogonal, i.e., $\mu(\omega)\mu(\omega') = 0$ for $\omega \neq \omega'$.

5. **Observables (for physicists only):** In this problem we discuss the relationship between projective measurements and 'observables' as introduced in a basic quantum mechanics class. An *observable* on a quantum system $X$ is by definition a Hermitian operator on $\mathcal{X}$.

   (a) Let $\mu\colon \Omega \to \mathrm{Pos}(\mathcal{X})$ be a *projective* measurement with $\Omega \subseteq \mathbb{R}$. Show that

   $$O = \sum_{\omega \in \Omega} \omega\mu(\omega) \tag{1}$$

   is an observable. Conversely, show that any observable can be written as in Eq. (1) for a suitable projective measurement $\mu$.

(b) What are the eigenvalues and eigenspaces of $O$?

(c) Now suppose that the system is in state $\rho$ and we perform the measurement $\mu$. Show that the expectation value of the measurement outcome is given by $\operatorname{tr}[\rho O]$. For a pure state $\rho = |\psi\rangle\langle\psi|$, this can also be written as $\langle\psi|O|\psi\rangle$. Do you recognize these formulas from your quantum mechanics class?

(d) Let $Y$ be another quantum system. Show that $O \otimes I_Y$ is an observable on the joint system and argue that it is associated to the measurement $\mu \otimes I_Y$ defined in class.