Symmetry and Quantum Information	February 26, 2018
Problem Set 3	
Michael Walter, University of Amsterdam	*due March 6, 2018*

Problem 1 (Irreducible representation of S_3, 2 points).
In Lecture 5, we discussed that $\mathcal{H} = \{\left(\begin{smallmatrix}\alpha\\ \beta\\ \gamma\end{smallmatrix}\right) \in \mathbb{C}^3 : \alpha + \beta + \gamma = 0\}$ is a representation of S_3, with the R_π acting by permuting the coordinates. Show that this representation is irreducible.

Problem 2 (Schur's lemma, 3 points).
In this problem, you can practice Schur's lemma. The two parts are independent of each other.

(a) Let $\mathcal{H}$ and $\mathcal{H}'$ be irreducible unitary representations and $J: \mathcal{H} \to \mathcal{H}'$ an intertwiner. Show that J is proportional to a unitary operator.

Hint: Show that $J^\dagger$ is also an intertwiner.

This strengthens part (i) of Schur's lemma, which asserted that either $J = 0$ or J is invertible.

(b) Let G be a commutative group (i.e., $gh = hg$ for all g, $h \in G$). Show that any irreducible representation of G is necessarily one-dimensional.

Problem 3 (Symmetries imply normal form, 3 points).
In this problem, you will show that quantum states that commute with U or $U^{\otimes 2}$ are tightly constrained by these symmetries.
First, recall that the single-qubit Hilbert space $\mathbb{C}^2$ is an irreducible representation of $U(2)$.

(a) Show that if ρ is a density operator on $\mathbb{C}^2$ such that $[\rho, U] = 0$ for every unitary $U \in U(2)$, then $\rho = I/2$.

From class you know that the two-qubit Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is not irreducible, but decomposes into two irreducible representations of $U(2)$: the symmetric subspace and a one-dimensional representation spanned by the singlet $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$.

(b) Show that if ρ is a density operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ such that $[\rho, U^{\otimes 2}] = 0$ for every $U \in U(2)$, then there exists $p \in [0,1]$ such that

$$\rho = p\,\tau_{\text{triplet}} + (1-p)\tau_{\text{singlet}}.$$

Here, $\tau_{\text{triplet}} = \Pi_2/3$, $\tau_{\text{singlet}} = |\Psi^-\rangle\langle\Psi^-|$. As always, Π_2 denotes the projector onto $\text{Sym}^2(\mathbb{C}^2)$.

Hint: Use Schur's lemma.

Problem 4 (Post-measurement state for density operators, 3 points).
Consider a quantum system described by an ensemble of pure quantum states $\{p_i, |\psi_i\rangle\}$, with corresponding density operator ρ. Suppose that we perform a projective measurement $\{P_x\}_{x\in\Omega}$ on the system. In this problem, you will derive a description of the post-measurement states.

(a) Verify that $\text{tr}[\rho P_x]$ equals the probability that the measurement outcome is x.

(b) Given that the outcome is x, compute the probability that the original state was $|\psi_i\rangle$.

Hint: Use Bayes' theorem.

(c) Given that the outcome is x, determine the ensemble of post-measurement states, and verify that the corresponding density operator is $P_x \rho P_x / \text{tr}[\rho P_x]$.