Problem Set 1

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Problem 1 (The ebit is entangled, 3 points).
Let $|\Psi\rangle = \sum_{i,j} M_{i,j} |i\rangle \otimes |j\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ be an arbitrary quantum state, expanded in the computational basis. Let M denote the $d \times d$-matrix with entries $M_{i,j}$.

(a) Show that $|\Psi\rangle = |\phi\rangle \otimes |\psi\rangle$ for some $|\phi\rangle, |\psi\rangle \in \mathbb{C}^d$ if and only if the rank of M is one.

(b) Conclude that the ebit state $|\Phi^+\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$ is entangled, as we claimed in class.

Problem 2 (Order of measurements, 4 points).
In this problem, you will see how the order of measurements can matter in quantum mechanics. Let $|\psi\rangle$ be an arbitrary state of a qubit.

(a) Imagine that we first measure the Pauli matrix X, with outcome x, and then the Pauli matrix Z, with outcome z. Derive a formula for the joint probability, denoted $p(x \to z)$, of the two measurement outcomes.

(b) Derive a similar formula for the joint probability $p(x \leftarrow z)$ corresponding to first measuring Z and then X.

(c) Find a state $|\psi\rangle$ such that $p(x \to z) \neq p(x \leftarrow z)$.

Problem 3 (Entanglement swapping, 4 points).
In class, we briefly discussed what happens when we teleport half of an entangled state. In this exercise, you will study this situation more carefully.

(a) Let $|\psi\rangle_{ME}$ be an arbitrary quantum state and consider the state $|\psi\rangle_{ME} \otimes |\Phi^+\rangle_{AB}$. Suppose that the M and A subsystems are in Alice' laboratory and the B subsystem is in Bob's laboratory, so that they can apply the teleportation protocol as in class. (Neither Alice nor Bob have access to the E subsystem.) Show that after completion of the teleportation protocol, the state of the B and E subsystems is $|\psi\rangle_{BE}$.

(b) Now assume that we have three nodes – Alice, Bob, and Charlie – such that Alice and Bob as well as Bob and Charlie start out by sharing an ebit each, i.e., the initial state is $|\Phi^+\rangle_{AB_1} \otimes |\Phi^+\rangle_{B_2C}$. Using teleportation as in (a), how can they establish an ebit between Alice and Charlie?

(c) Sketch how to extend the scheme in (b) to a linear chain of N nodes, assuming that initially only neighboring nodes share ebits.

Problem 4 (Distinguishing quantum states, 6 points).
The *trace distance* between two quantum states $|\phi\rangle$ and $|\psi\rangle$ is defined by

$$T(\phi, \psi) = \max_{0 \leq Q \leq I} \langle\phi|Q|\phi\rangle - \langle\psi|Q|\psi\rangle. \tag{1.1}$$

Here, $0 \leq Q \leq I$ means that both Q and $I - Q$ are positive semidefinite operators.

(a) Imagine a quantum source that emits $|\phi\rangle$ or $|\psi\rangle$ with probability 1/2 each. Show that the optimal probability of identifying the true state by a POVM measurement is given by

$$\frac{1}{2} + \frac{1}{2}T(\phi, \psi).$$

Without using this formula: Why can this probability never be smaller than 1/2?

(b) Conclude that only orthogonal states (i.e., $\langle\phi|\psi\rangle = 0$) can be distinguished perfectly.

(c) Show that the trace distance is a metric. That is, verify that $T(\phi, \psi) = 0$ if and only if $|\phi\rangle = e^{i\theta}|\psi\rangle$, that $T(\phi, \psi) = T(\psi, \phi)$, and prove the triangle inequality $T(\phi, \psi) \leq T(\phi, \chi) + T(\chi, \psi)$.

You will now derive an explicit formula for the trace distance. For this, consider the spectral decomposition $\Delta = \sum_i \lambda_i |e_i\rangle\langle e_i|$ of the Hermitian operator $\Delta = |\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|$.

(d) Show that the operator $Q = \sum_{\lambda_i > 0} |e_i\rangle\langle e_i|$ achieves the maximum in (1.1), and deduce the following formulas for the trace distance:

$$T(\phi, \psi) = \sum_{\lambda_i > 0} \lambda_i = \frac{1}{2}\sum_i |\lambda_i|.$$

(e) Conclude that the optimal probability of distinguishing the two states in (a) remains unchanged if we restrict to projective measurements.

In class, we will also use the *fidelity* $|\langle\phi|\psi\rangle|$ to compare quantum states.

(f) Show that trace distance and fidelity are related by the following formula:

$$T(\phi, \psi) = \sqrt{1 - |\langle\phi|\psi\rangle|^2}.$$

Hint: Argue that it suffices to verify this formula for two pure states of a qubit, with one of them equal to $|0\rangle$. Then use the formula from part (d).

This exercise shows that states with fidelity close to one are almost indistinguishable by any measurement.

Problem 5 (POVMs can outperform proj. measurements, 4 points; Nielsen & Chuang §2.2.6)**.** Imagine a qubit source that emits either of the two states $|0\rangle$ and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ with equal probability 1/2. Your task is to design a measurement that optimally distinguishes these two cases. Unfortunately, the states $|0\rangle$ and $|+\rangle$ are not orthogonal, so you know that this cannot be done perfectly (e.g., from the previous problem).

Suppose now that your measurement is allowed to report one of *three* possible outcomes: that the true state is $|0\rangle$, that the true state is $|+\rangle$, or that the measurement outcome is inconclusive. However, it is *not allowed to ever give a wrong answer*! We define the success probability of such a measurement scheme as the probability that you identify the true state.

(a) Show that for projective measurements the success probability is at most 1/4.

(b) Find a POVM measurement that achieves a success probability strictly larger than 1/4.