Yesterday: Density operators and partial trace

Saw that ρ_{AB} pure can have ρ_A mixed. Conversely:

Any density operator ρ_A has purification $|\psi_{AB}\rangle$:

$$\mathrm{tr}_B\left[|\psi_{AB}\rangle\langle\psi_{AB}|\right] = \rho_A$$

* Mixed states $\sim$ subsystem of larger pure states
  (cf. observables vs. POVMs)

CHURCH OF LARGER H.SPACE

* Existence: If $\rho_A = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ spectral decomp.

$\hookrightarrow$ $|\psi_{AB}\rangle = \sum_i \sqrt{p_i}\, |e_i\rangle_A \otimes |i\rangle_B$ works

* Uniqueness: If $|\psi_{AB}\rangle, |\tilde{\psi}_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ are two purifications, $\exists$ unitary U_B:

$$\boxed{(I_A \otimes U_B)|\psi_{AB}\rangle = |\tilde{\psi}_{AB}\rangle}$$

Exercise

Schmidt decomposition: Any $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as

$$|\psi_{AB}\rangle = \sum_i \underbrace{s_i}_{>0} |e_i\rangle \otimes |f_i\rangle$$

(SVD!)

ortho-normal

ortho-normal

reduced density matrices

$$\rho_A = \sum_i |s_i|^2 \, |e_i\rangle\langle e_i|_A \qquad \rho_B = \sum_i |s_i|^2 \, |f_i\rangle\langle f_i|_B$$

eigenvalues are same

* s_i are unique

* $|\psi_{AB}\rangle$ product state $\Longleftrightarrow$ ρ_A pure $\Longleftrightarrow$ ρ_B pure

  i.e. $|\psi_{AB}\rangle$ entangled $\Longleftrightarrow$ ρ_A mixed $\Longleftrightarrow$ ρ_B mixed

  ↳ $\{s_i\}$ characterize entanglement for pure states $\boxed{\text{PSET 4}}$

Important consequence: For every ρ_{AB}:

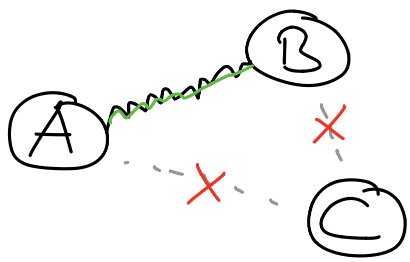
$$\rho_A \text{ pure} \Longrightarrow \rho_{AB} = \rho_A \otimes \rho_B$$

remember the GHZ game: if we know state is $|\Gamma\rangle$, must be $\otimes$ with everyone else

Pf: Let $|\psi_{ABC}\rangle$ purification of ρ_{AB}.

ρ_A pure $\overset{\text{above}}{\Longrightarrow}$ $|\psi_{ABC}\rangle = |\psi_A\rangle \otimes |\psi_{BC}\rangle$ $\overset{\text{partial}}{\underset{\text{trace}}{\Longrightarrow}}$ $\rho_{AB} = \rho_A \otimes \rho_B$ $\square$

Example:



MONOGAMY

$$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$$

$$\Downarrow$$

$$\rho_{ABC} = \rho_{AB} \otimes \rho_C$$

$\Longrightarrow$ $\rho_{AC} = \rho_A \otimes \rho_C$
& $\rho_{BC} = \rho_B \otimes \rho_C$

Can we make this more quantitative?

When is a q. state <u>entangled</u>?

For pure states: $|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$

In general: ρ_{AB} <u>entangled</u> if   if ρ_{AB} pure

$$\rho_{AB} \neq \boxed{\sum_i p_i\, \rho_A^i \otimes \rho_B^i}$$

unentangled or "<u>separable</u>" state

- ensemble of product states
- e.g., any $\sum_{a,b} p(a,b)\, |a\rangle\langle a| \otimes |b\rangle\langle b|$ classical correl. $\neq$ entanglement

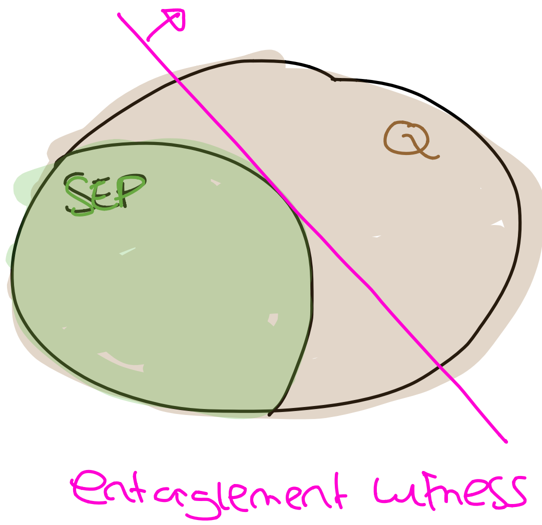
   joint probability distribution

- all states that can be created by local ops + classical commun. (LOCC)

- Convex geometry perspective:

$$\{\rho_{AB} \geq 0,\ \mathrm{tr}\,\rho_{AB} = 1\} = Q$$
$$\cup|$$
$$\{\rho_{AB} \text{ separable}\} =: SEP$$

Convex sets!



* NP-hard to test if given $\rho_{AB} \in SEP$!   PSET

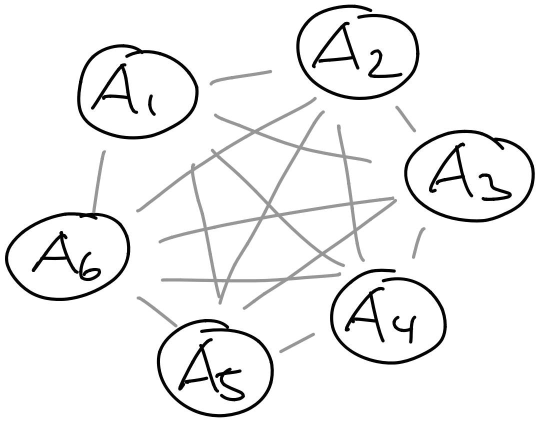
...but you will learn the best known test...

Monogamy & Symmetry

① In a permutation-symmetric state

$$|\Phi_{A_1 \dots A_N}\rangle \in \mathrm{Sym}^N(\mathbb{C}^d)$$

would expect that $\rho_{A_iA_j}$ "not very" entangled:



Ex: Mean-field $H = \sum_{i<j} h_{ij}$

same interaction

$\hookrightarrow |E_0\rangle \in \mathrm{Sym}^N$

if nondegenerate

Pset

Ex: n bosons

De Finetti Theorem:

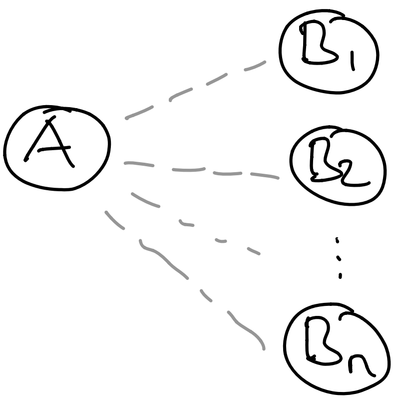
$$\rho_{A_1 \dots A_k} \approx \boxed{\int d\psi \, p(\psi) \, |\psi\rangle\langle\psi|^{\otimes k}}$$

mixture of tensor powers ($\Rightarrow$ separable)

Ex: $|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$ ⟵ entangled for any bipartition

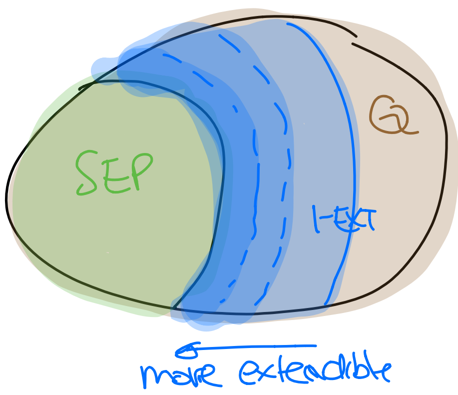
$$\Longrightarrow \rho_{A_1A_2} = \frac{1}{2}\left[|0\rangle\langle 0|^{\otimes 2} + |1\rangle\langle 1|^{\otimes 2}\right]$$ ⟵ unentangled 😊

② If a state ρ_{AB_1} can be extended to $\rho_{AB_1 \dots B_n}$ s.th. $\rho_{AB_1} = \dots = \rho_{AB_n}$ then ρ_{AB_1} "not very" entangled



$$\rho_{AB_1} \approx \boxed{\sum_i p_i \, \rho_A^i \otimes \rho_B^i}$$

separable



① Proof of the de Finetti theorem

$N = k + n$

$$|\Phi\rangle \in \mathrm{Sym}^{k+n}(\mathbb{C}^d)$$

Idea: Measure $\{Q_\psi = (\,)\,|\psi\rangle\langle\psi|^{\otimes n}\}$ on last n systems

$\longrightarrow$ first k should be $|\psi\rangle^{\otimes k}$

Set $\rho := |\Phi\rangle\langle\Phi|$. Then: NOTATION!!!

$$\rho_k = \mathrm{tr}_n[|\Phi\rangle\langle\Phi|] = \mathrm{tr}_n[(I_k \otimes \Pi_n)|\Phi\rangle\langle\Phi|]$$

$$= \binom{n+d-1}{n} \int d\psi \; \mathrm{tr}_n[(I_k \otimes |\psi\rangle\langle\psi|^{\otimes n})|\Phi\rangle\langle\Phi|]$$

$$\overset{!!!}{=} \binom{n+d-1}{n} \int d\psi \; (I_k \otimes \langle\psi|^{\otimes n})|\Phi\rangle\langle\Phi|(I_k \otimes |\psi\rangle^{\otimes n})$$

$$= \int d\psi \; p(\psi)\, |V_\psi\rangle\langle V_\psi|$$

where we defined

$$\sqrt{p(\psi)}\, |V_\psi\rangle := \binom{n+d-1}{n}^{1/2} (I_k \otimes \langle\psi|^{\otimes n})|\Phi\rangle$$

$$\in \mathrm{Sym}^k(\mathbb{C}^d)$$

prob. density

unit vector

Claim: $\rho_k = \int d\psi\, p(\psi)\, |V_\psi\rangle\langle V_\psi| \approx \rho_k' := \int d\psi\, p(\psi)\, |\psi\rangle^{\otimes k}\langle\psi|^{\otimes k}$

Average overlap squared:

$$\int d\psi \, p(\psi) \, |\langle \psi^{\otimes k} | V_\psi \rangle|^2$$

$$= \binom{n+d-1}{n} \int d\psi \, |\langle \psi^{\otimes(n+k)} | \Phi \rangle|^2$$

$$= \binom{n+d-1}{n} \underline{\int d\psi} \, \langle \Phi | \underline{|\psi\rangle^{\otimes(n+k)} \langle \psi|^{\otimes(n+k)}} | \Phi \rangle$$

i magic formula!

$$= \binom{n+d-1}{n} \underline{\binom{n+k+d-1}{n+k}^{-1}} \underbrace{\langle \Phi | \underline{\Pi_{n+k}} | \Phi \rangle}_{=1}$$

lecture 4

$$\geq 1 - \frac{dk}{n}$$

SUCCESS IF $dk \ll n$

e.g. k fix & $n \to \infty$ (thermodyn. limit)

To make more precise: Use <u>trace distance</u>

$$T(\rho, \sigma) = \max_{0 \leq Q \leq I} \left(\mathrm{tr}[Q\rho] - \mathrm{tr}[Q\sigma] \right) = \frac{1}{2} \|\rho - \sigma\|_1$$

trace norm:

$$\|\Delta\|_1 = \sum_i |\lambda_i|$$

if $\Delta = \sum_i \lambda_i |e_i\rangle\langle e_i|$

If $\rho = |\psi\rangle\langle\psi|$, $\sigma = |\phi\rangle\langle\phi|$:

$$T(\rho, \sigma) = \sqrt{1 - |\langle \psi | \phi \rangle|^2}$$

overlap squared

$\boxed{\text{PSET 1}}$

Thus:

$$T(\rho_k, \rho'_k) = \frac{1}{2} \left\| \int d\psi \, p(\psi) \left(|v_\psi\rangle\langle v_\psi| - |\psi\rangle\langle\psi|^{\otimes k} \right) \right\|_1$$

$$\leq \int d\psi \, p(\psi) \, \frac{1}{2} \left\| \, |v_\psi\rangle\langle v_\psi| - |\psi\rangle\langle\psi|^{\otimes k} \right\|_1$$

$$= \int d\psi \, p(\psi) \sqrt{1 - |\langle v_\psi | \psi^{\otimes k}\rangle|^2}$$

$$\overset{\text{Jensen}}{\leq} \sqrt{\int d\psi \, p(\psi) \left(1 - |\langle v_\psi | \psi^{\otimes k}\rangle|^2\right)}$$

$$\leq \sqrt{1 - \left(1 - \frac{dk}{n}\right)} = \sqrt{\frac{dk}{n}}$$

$\square$