Yesterday: Intro to representation theory. Highlights:

* Part (2) of Schur's lemma: If $\mathcal{H}$ is an irrep & $J:\mathcal{H}\to\mathcal{H}$ intertwiner: $\boxed{J \propto I_{\mathcal{H}}}$ $\quad$ $\oplus$ of reps

* Claim: $\mathrm{Sym}^n(\mathbb{C}^d)$ is not only invariant subspace, but in fact irreducible repr. of (S)U(d).

Consequence: $\Pi_n = \Pi_n' := \binom{n+d-1}{n} \int d\psi\, |\psi\rangle^{\otimes n}\langle\psi|^{\otimes n}$

Proof: Let $\mathcal{H} := \mathrm{Sym}^n(\mathbb{C}^d)$.

With respect to $(\mathbb{C}^d)^{\otimes n} = \mathcal{H} \oplus \mathcal{H}^\perp$: $\quad$ restriction of $U^{\otimes n}$ to $\mathcal{H}$

$$\Pi_n = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right), \quad \Pi_n' =: \left(\begin{array}{c|c} J & 0 \\ \hline 0 & 0 \end{array}\right), \quad T_U = U^{\otimes n} = \left(\begin{array}{c|c} T_U^{\mathcal{H}} & 0 \\ \hline 0 & T_U^{\mathcal{H}^\perp} \end{array}\right)$$

restriction of $U^{\otimes n}$ to $\mathcal{H}^\perp$

Invariance of the measure $d\psi$ implies:

$$U^{\otimes n}\Pi_n' = \Pi_n' U^{\otimes n} \qquad \left(\begin{array}{c|c} T_U^1 & \\ \hline & T_U^2 \end{array}\right)\left(\begin{array}{c|c} J & \\ \hline & \end{array}\right) =$$

$\Rightarrow T_U^{\mathcal{H}} J = J\, T_U^{\mathcal{H}}$, i.e. J is intertwiner $\mathcal{H} \to \mathcal{H}$

$\mathcal{H}$ is irreducible $\xRightarrow[\text{lemma}]{\text{Schur's}}$ $J = \lambda \cdot I_{\mathcal{H}} \Rightarrow \Pi_n' = \lambda \cdot \Pi_n$ (for some λ)

To see that $\lambda = 1$, compare traces:

$$\mathrm{tr}[\Pi_n'] = \binom{n+d-1}{n} \underbrace{\int d\psi\, \underbrace{\mathrm{tr}[|\psi\rangle\langle\psi|]^n}_{=1}}_{=1} = \binom{n+d-1}{n}$$

$$\mathrm{tr}[\lambda \cdot \Pi_n] = \lambda \cdot \mathrm{tr}[\Pi_n] = \lambda \cdot \dim \mathcal{H} = \lambda \cdot \binom{n+d-1}{n} \quad \square$$

Aside: Recall that We did not discuss this in class.

$\mathcal{H} := \mathrm{Sym}^n(\mathbb{C}^d)$ is inv. subspace for both $U(d)$ and S_n:

$$T_U = U^{\otimes n} = \left(\begin{array}{c|c} T_U^{\mathcal{H}} & 0 \\ \hline 0 & T_U^{\mathcal{H}^\perp} \end{array}\right), \quad R_\pi = \left(\begin{array}{c|c} R_\pi^{\mathcal{H}} & 0 \\ \hline 0 & R_\pi^{\mathcal{H}^\perp} \end{array}\right)$$

$$[T_U, R_\pi] = 0 \implies \boxed{T_U^{\mathcal{H}} R_\pi^{\mathcal{H}} = R_\pi^{\mathcal{H}} T_U^{\mathcal{H}}}$$

* $R_\pi^{\mathcal{H}}$ is intertwiner for $U(d)$-irrep $\mathcal{H}$

  $\overset{\text{Schur}}{\implies} R_\pi^{\mathcal{H}} \propto I_{\mathcal{H}}$. Indeed, $R_\pi^{\mathcal{H}} = I_{\mathcal{H}}$!

  "abstract matches concrete"

* $T_U^{\mathcal{H}}$ is intertwiner for S_n-repr $\mathcal{H}$, but:

  NOT an irrep, Schur does NOT apply

Proof that symmetric subspace is irrep of (S)U(d)

Here: $d=2$ d>2 works similarly; Can you do it?

Warmup: $\mathrm{Sym}^n(\mathbb{C}^2)$ has basis

$$|\omega_{n,0}\rangle = |0\cdots 0\rangle = |0\rangle^{\otimes n}$$

$$|\omega_{n-1,1}\rangle \propto |\underbrace{0..0}_{n-1}1\rangle + |\underbrace{0..0}_{n-2}10\rangle + \dots + |1\underbrace{0...0}_{n-1}\rangle$$

⋮

$|\omega_{m,n-m}\rangle \propto |0\rangle^{\otimes m}\, |1\rangle^{\otimes(n-m)}$ + permutations

⋮

$|\omega_{0,n}\rangle = |1\ldots1\rangle = |1\rangle^{\otimes n}$

"occupation numbers"

Why? Apply Π_n to computational basis.

$\# = n+1 = \binom{n+2-1}{n}$ ✓

Strategy: If $\tilde{\mathcal{H}} \neq \{0\}$ inv. subspace of $\mathrm{Sym}^n(\mathbb{C}^d)$, show

① $\tilde{\mathcal{H}} \ni$ some $|\omega_{m,n-m}\rangle$ ② $\tilde{\mathcal{H}} \ni$ all $|\omega_{m,n-m}\rangle$

Idea: Convert into a linear algebra problem... "Lie algebra", "generators", ...

* For every operator M on $\mathbb{C}^2$, define

$$\tilde{M} = M \otimes I \otimes \cdots \otimes I + \ldots + I \otimes \cdots \otimes I \otimes M =: \sum_{k=1}^{n} M_k$$

on $(\mathbb{C}^2)^{\otimes n}$. Why?

* $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Longrightarrow \tilde{Z}\,|i_1 \ldots i_n\rangle = (\#0\text{'s} - \#1\text{'s})\,|i_1 \ldots i_n\rangle$

$$\tilde{Z}\,\boxed{|\omega_{m,n-m}\rangle} = (m-(n-m))\,|\omega_{m,n-m}\rangle$$
$$= (2m-n)\,|\omega_{m,n-m}\rangle$$

eigenvectors w/ distinct eigenvalues

* $M_+ := |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $M_- := |1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\hookrightarrow \tilde{M}_+ |\omega_{m,n-m}\rangle \propto |\omega_{m+1,n-m-1}\rangle$ if $m<n$

$\tilde{M}_- |\omega_{m,n-m}\rangle \propto |\omega_{m-1,n-m+1}\rangle$ if $m>0$

So we see that $\tilde{M}$ preserve the symm. subspace... in fact

What is the <u>meaning</u> of the $\tilde{M}$'s?

* $U(d) = \{ \boxed{e^{iM}} \mid M = M^\dagger \}$

Matrix exponential: $e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$

If $A = \sum_i a_i |\phi_i\rangle\langle\phi_i| \Rightarrow$
$e^A = \sum_i e^{a_i} |\phi_i\rangle\langle\phi_i|$

① $(e^A)^\dagger = e^{(A^\dagger)}$

② $e^{A\otimes I} = e^A \otimes I$

③ $[A,B]=0 \Rightarrow e^A e^B = e^{A+B}$

④ $U e^A U^\dagger = e^{UAU^\dagger}$

⑤ $\det(e^A) = e^{\mathrm{tr}[A]}$

$\ulcorner$ So if $M=M^\dagger$: $e^{iM}(e^{iM})^\dagger \overset{①}{=} e^{iM} e^{-iM} \overset{③}{=} e^{iM-iM} = e^0 = I \lrcorner$

* Thus:

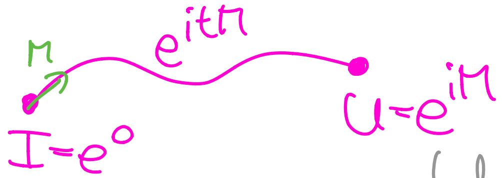
$$e^{i\tilde{M}} = e^{i\sum_k M_k} \overset{③}{=} e^{iM_1} \cdots e^{iM_k}$$

$$\overset{②}{=} (e^{iM} \otimes I \otimes \cdots \otimes I) \cdots (I \otimes \cdots \otimes I \otimes e^{iM})$$

$$= (e^{iM})^{\otimes n}$$

If $M=M^\dagger$: $\boxed{U:=e^{iM} \Longrightarrow U^{\otimes n} = e^{i\tilde{M}}}$

$$M = -i\partial_{t=0}\left[e^{itM}\right]$$



$$\tilde{M} = -i\partial_{t=0}\Big[\underbrace{e^{it\tilde{M}}}_{(e^{itM})^{\otimes n}}\Big]$$

"Lie algebra representation"

$$\begin{tikzcd}
U \arrow[r, mapsto] \arrow[dr, phantom, "///"] & U^{\otimes n} \\
M \arrow[r, mapsto] \arrow[u, "e^{iM}"] & \tilde{M} \arrow[u, "e^{i\tilde{M}}"']
\end{tikzcd}$$

Now let $\mathcal{H} \subseteq \mathrm{Sym}^n(\mathbb{C}^2)$ inv. subspace for $U(2)$:

* Claim: For all M: $\boxed{\tilde{M}\mathcal{H} \subseteq \mathcal{H}}$

Pf: WLOG $M=M^\dagger$. If $|\psi\rangle \in \mathcal{H}$ then

vector spaces are closed

$$\tilde{M}|\psi\rangle = -i\partial_{t=0}\boxed{e^{it\tilde{M}}|\psi\rangle} \in \mathcal{H}.$$

$= (e^{itM})^{\otimes n}|\psi\rangle \in \mathcal{H}$ since inv. subspace $\square$

* If $\mathcal{H} \neq \{0\}$:

$\tilde{Z}\mathcal{H} \subseteq \mathcal{H} \Rightarrow \mathcal{H}$ spanned by eigenvectors

$\Rightarrow \mathcal{H} \ni$ some $|\omega_{m,n-m}\rangle$

$\tilde{M}_\pm \mathcal{H} \subseteq \mathcal{H} \Rightarrow \mathcal{H} \ni$ all $|\omega_{m,n-m}\rangle$

$\Rightarrow \mathcal{H} = \mathrm{Sym}^n(\mathbb{C}^2)$

PHEW! $\square$