<u>Last time</u>: Pure state estimation via $\mathrm{Sym}^n(\mathbb{C}^d)$.

$$\Pi_n \overset{!}{=} \binom{d+n-1}{n} \int d\psi \; |\psi\rangle^{\otimes n}\langle\psi|^{\otimes n} =: \Pi_n^!$$

orthogonal proj. onto Sym. subspace

dimension of Sym. subspace

prob. measure on pure states $\{\psi = |\psi\rangle\langle\psi|\}$, inv. under $\psi \mapsto U|\psi\rangle\langle\psi|U^\dagger = U\psi U^\dagger$

$$\boxed{U^{\otimes n}\,\Pi_n^!\,(U^\dagger)^{\otimes n} = \Pi_n^!}$$

<u>Today</u>: Will prove this formula using representation theory

(Alternative: Calculate the integral "by hand".)

<u>Literature</u>: Part I in Serre $\overset{!}{\;}$ $\to$ course homepage

<u>Group</u> G: Set with multiplication ("·"), neutral element ("1"), inverses ("g^{-1}")

$g,h \in G \Rightarrow g\cdot h \in G \qquad g \in G \Rightarrow g\cdot 1 = 1\cdot g = g$

$g \in G \Rightarrow \exists g^{-1}: g\cdot g^{-1} = g^{-1}\cdot g = 1 \qquad$ Often omit "·"!

<u>Examples</u>:

* Symmetric group $S_n := \{\pi : \{1,\dots,n\} \to \{1,\dots,n\}$ permutation$\}$

  · = composition: $(\pi\cdot\tau)(x) := \pi(\tau(x))$

  1 = identity map, π^{-1} = inverse function

* Unitary group $U(d) := \{U \text{ unitary } d\times d \text{ matrix}\}$

$\cdot$ = matrix multipl., $1 = I$ = identity matrix, $U^{-1} = U^\dagger$

* Special unitary group $SU(d) = \{U \text{ unitary} \mid \det(U) = 1\}$

↳ subgroup of $U(d)$

Other groups? D_8, $\mathbb{Z}/n\mathbb{Z}$, $GL(d)$ & $SL(d)$, ...

Introduction to Representation Theory

Unitary representation of group G:

* Hilbert space $\mathcal{H}$

* unitary operators $\{R_g\}_{g\in G}$ on $\mathcal{H}$ s.th.

$$R_{gh} = R_g \cdot R_h \quad \& \quad R_1 = I_{\mathcal{H}}$$

NB: Always $\dim < \infty$. Will say "the representation $\mathcal{H}$".

Examples:

* $(\mathbb{C}^d)^{\otimes n}$ is repr. of S_n and of $U(d)$

$$R_\pi(|\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle) = |\psi_{\pi^{-1}(1)}\rangle \otimes \dots \otimes |\psi_{\pi^{-1}(n)}\rangle$$

$$T_U = U^{\otimes n} = U \otimes \dots \otimes U$$

$$\boxed{[R_\pi, T_U] = 0}$$!

... so we can think of repr. of $S_n \times U(d)$

$*$ Representations of S_3: $\left\{\begin{matrix} \mathrm{id}, 1\leftrightarrow 2, 1\leftrightarrow 3, 2\leftrightarrow 3, \\ 1\to 2\to 3, 3\to 2\to 1 \end{matrix}\right\}$

- <u>Trivial</u> repr: Exists for any group.

$$\mathcal{H} = \mathbb{C}|0\rangle \qquad R_\pi |0\rangle = |0\rangle \quad \forall \pi$$

- <u>Sign repr</u>:

$$\mathcal{H} = \mathbb{C}|0\rangle \qquad R_\pi |0\rangle = \mathrm{Sign}(\pi)\, |0\rangle$$

-1 for swaps $1\leftrightarrow 2$ etc.

- $\mathcal{H} = \mathbb{C}^3 = \{\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle\}$ $\circledast$

R_π permutes coords, e.g. $R_{1\leftrightarrow 2}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}$

To understand a representation, want to decompose it into its smallest building blocks...

<u>Invariant subspace</u> ("subrepresentation"): $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ s.th.

$$|\psi\rangle \in \tilde{\mathcal{H}} \Rightarrow R_g|\psi\rangle \in \tilde{\mathcal{H}} \quad (\forall g)$$

$*$ $\mathcal{H}$ is called <u>irreducible</u> ("irrep") if $\{0\}$ & $\mathcal{H}$ are only invariant subspaces. elementary building block

$*$ If $\tilde{\mathcal{H}}$ is inv. subspace so is $\tilde{\mathcal{H}}^\perp$.

$$\mathcal{H} = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}^\perp$$

$$R_g = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

"Smaller" representations

nonempty if $\tilde{\mathcal{H}} \neq \{0\}, \mathcal{H}$

$\leadsto$ "finest" <u>decomposition</u>: $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$

orthogonal & irreducible

RT tells us how to decompose & what the irreps are!

<u>Example</u>:

* Any 1-dim. repr. is an irrep.

* (*) is NOT an irrep of S_3, because

$$\tilde{\mathcal{H}} = \{\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \mid \alpha+\beta+\gamma=0\} \subseteq \mathbb{C}^3$$

is inv. subspace. $\tilde{\mathcal{H}}$ is irreducible. $\rightarrow$ PSET!

$$\tilde{\mathcal{H}}^\perp = \mathbb{C}(|0\rangle + |1\rangle + |2\rangle) \text{ is 1-dim}$$

$\hookrightarrow$ $\mathcal{H} = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}^\perp$ is decomposition into irreps

* $\mathrm{Sym}^n(\mathbb{C}^d)$?

Any subspace is invariant!

For S_n, invariant but <u>not</u> irreducible.

For $U(d)$, $\boxed{\text{invariant}}$ <u>and</u> IRREDUCIBLE!

$R_\pi|\Phi\rangle = |\Phi\rangle \Rightarrow R_\pi(T_u|\Phi\rangle) = T_u(R_\pi|\Phi\rangle) = T_u|\Phi\rangle$

Proof below for $d=2$.

Intertwiner: $J : \mathcal{H} \to \mathcal{H}'$ s.th. $J R_g = R'_g J \ (\forall g)$

If J is unitary: $\mathcal{H}, \mathcal{H}'$ are called (unitarily) equivalent

$$\boxed{J R_g J^\dagger = R'_g}$$ base change

" $\mathcal{H} \cong \mathcal{H}'$ "

Schur's Lemma: Let $J : \mathcal{H} \to \mathcal{H}'$ intertwiner.

1. If $\mathcal{H}, \mathcal{H}'$ irreps: J invertible or $J = 0$.
2. If $\mathcal{H} = \mathcal{H}'$ same irrep: $\boxed{J \propto I_{\mathcal{H}}}$

Proof: 1. $\ker(J)$ & $\operatorname{ran}(J)$ are inv. subspaces

2. Let λ be an eigenvalue. Then $\ker(\lambda - J) \neq 0$, so $\ker(\lambda - J) = \mathcal{H}$, so $J = \lambda \cdot I_{\mathcal{H}}$. $\square$

WHY DO WE CARE?!

Consequence: $\Pi_n = \Pi'_n := \binom{n+d-1}{n} \int d\psi \, |\psi\rangle^{\otimes n} \langle \psi|^{\otimes n}$

Sketch: W.r.t. $(\mathbb{C}^d)^{\otimes n} = \mathcal{H} \oplus \mathcal{H}^\perp$, $\mathcal{H} := \operatorname{Sym}^n(\mathbb{C}^d)$.

$$\Pi_n = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right), \quad \Pi'_n = \left(\begin{array}{c|c} J & 0 \\ \hline 0 & 0 \end{array}\right), \quad U^{\otimes n} = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

$U^{\otimes n} \Pi'_n (U^\dagger)^{\otimes n} = \Pi'_n \Rightarrow J$ is intertwiner on irrep

Schur's lemma $\Rightarrow J \propto I \Rightarrow \Pi_n' \propto \Pi_n$

To see that "=", compare trace. $(\square)$

<u>Next time</u>: More details on this

+ proof that $\mathrm{Sym}^n(\mathbb{C}^2)$ is irreducible.