This course:

Intro to $\boxed{\text{QIT}}$ from the perspective of $\boxed{\text{symmetries}}$

interdisciplinary field

goal is to leverage QM to process information

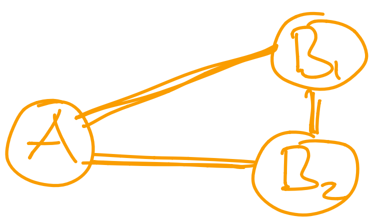
storage, transmission, processing of Q. information; design of high-precision measurements; Q. cryptography, computation

language/toolbox

"correlation", "qubit", "information", "entropy", "complexity", "computation", ...

QIT is "synthetic", why symmetries?

$|\psi\rangle^{\otimes n}$ many copies $\longrightarrow$ perm. sym.



monogamy of entanglement

$S(\rho)=S(U\rho U^\dagger)$ entropy $\longrightarrow$ unitary sym.

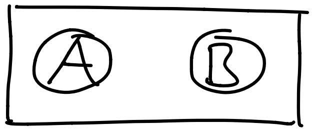
On the technical level, symmetries enter through:

Commut. relations, group actions, representation theory

Introduction to Quantum Mechanics

Axioms are a 1^{st} attempt – we will improve them soon!

$\boxed{\text{Axiom (A): To every q. system, we associate a Hilbert Space } \mathcal{H}. \text{ For a joint system composed of subsystems with Hilbert spaces } \mathcal{H}_A, \mathcal{H}_B \text{ the Hilbert space is } \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.}$



* fin-dim HS = vector space & inner product $\langle - | - \rangle$
* Throughout this course: $\dim \mathcal{H} < \infty$
* Simplest system: qubit $\mathbb{C}^2$

  ↳ n qubits: $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = (\mathbb{C}^2)^{\otimes n}$ $\quad \dim = 2^n$

$\boxed{\text{Axiom (B): Unit vectors } |\psi\rangle \in \mathcal{H} \text{ describe the state of a system.}}$

* Dirac notation: $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix}$ "ket"

$X^\dagger = \overline{X}^T$ adjoint

$\langle \psi | = |\psi\rangle^\dagger = (\overline{\psi_1} \;\; \overline{\psi_2} \;\; \cdots)$ "bra"

$\langle \psi | \phi \rangle = \sum_i \overline{\psi_i} \phi_i$ "bra-ket" $\quad \langle \psi | \psi \rangle = \| \psi \|^2$

anti-linear in 1^{st} argument!!!

* Standard ("Computational") basis:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{of } \mathbb{C}^2$$

A BIT

$$|i_1, i_2, \ldots, i_n\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_n\rangle \quad \text{of } (\mathbb{C}^2)^{\otimes n}$$

* NOT every state is a tensor product !!!

e.g. $\mathbb{C}^2 \otimes \mathbb{C}^2$:

PSET

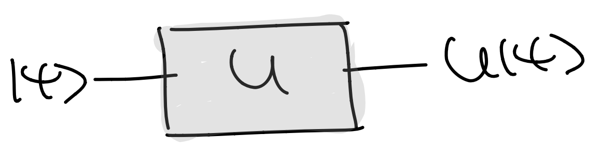
$$|\Phi^+\rangle := \sqrt{\tfrac{1}{2}}\left(|0,0\rangle + |1,1\rangle\right) \neq |\psi\rangle \otimes |\phi\rangle$$

maximally entangled state, ebit, EPR pair

Def: $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is called entangled if

$|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\phi_B\rangle$ for all $|\psi_A\rangle \in \mathcal{H}_A$, $|\phi_B\rangle \in \mathcal{H}_B$

$$\boxed{\text{Axiom (C): Given a unitary matrix } U \text{ on } \mathcal{H}, \text{ the transformation } |\psi\rangle \longmapsto U|\psi\rangle \text{ is in principle physical.}}$$



* Schrödinger eqn?
$-i\partial_t|\psi\rangle = H|\psi\rangle$

* unitary means $UU^\dagger = U^\dagger U = I$ (identity matrix)

* $\|U|\psi\rangle\|^2 = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|\psi\rangle = \|\psi\|^2$

↳ unit vectors (states) are preserved

Axiom (D): Any Hermitian operator X on $\mathcal{H}$ corresponds to an observable quantity: Let $X = \sum_x x \cdot P_x$ be the spectral decomposition. Born's rule:

eigenvalues

projector onto eigenspaces

When system is in state ψ,

$$\Pr(\text{outcome } x) = \langle \psi | P_x | \psi \rangle$$

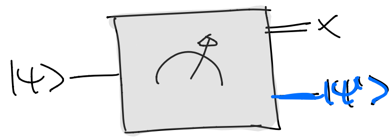
Thus:

$$E[\text{outcome}] = \sum_x x \langle \psi | P_x | \psi \rangle = \langle \psi | X | \psi \rangle$$

$P_x = |e_x\rangle\langle e_x|$ if non-deg.

Moreover, if the outcome is x then the post-meas. state is

$$|\psi'\rangle = \frac{P_x|\psi\rangle}{\| P_x |\psi\rangle \|} = \frac{P_x|\psi\rangle}{\sqrt{\langle \psi | P_x | \psi \rangle}}$$



* measurement "collapses" the state !!!

↳ fragility of q. info, "decoherence" ...

$P_x|\psi\rangle \perp P_{x'}|\psi\rangle$

* NO collapse $\Leftrightarrow |\psi'\rangle = |\psi\rangle \Leftrightarrow P_x|\psi\rangle = \delta_{x,x_0}|\psi\rangle$

$\Leftrightarrow$ $|\psi\rangle$ eigenvector of X $\Leftrightarrow \langle \psi | P_x | \psi \rangle = \delta_{x,x_0}$

$\Leftrightarrow$ outcome DETERMINISTIC

* When can we distinguish $|\psi\rangle, |\phi\rangle$ perfectly?

If orthogonal, i.e. $\langle\psi|\phi\rangle = 0$! Use observable ↳ $\boxed{\text{PSET}}$

$$X = |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \leftarrow \text{spectral decompo!!}$$

* $|\psi\rangle$ and $e^{i\vartheta}|\psi\rangle$ are completely indistinguishable

↳ $\rho = |\psi\rangle\langle\psi|$ density matrix characterizes state up to overall phase

Measuring a qubit

Infinitely many inequiv. measurements possible. $\boxed{\text{Cf. bit}}$

E.g.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |+\rangle\langle+| - |-\rangle\langle-|$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = |L\rangle\langle L| - |R\rangle\langle R|$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Spectral decompo

↳ eigenvalues ± 1

where $|+\rangle = \sqrt{\frac{1}{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \sqrt{\frac{1}{2}}(|0\rangle - |1\rangle)$

$|L\rangle = \sqrt{\frac{1}{2}}(|0\rangle + i|1\rangle)$, $|R\rangle = \sqrt{\frac{1}{2}}(|0\rangle - i|1\rangle)$

* $\{I, X, Y, Z\}$ is basis of Herm. matrices

* Pauli matrices do <u>not</u> commute

$$[X,Y] := XY - YX = 2iZ \neq 0 \quad \text{(etc.)}$$

↳ Order of measurement matters! ≙ "Joint measurement" does not make sense.

* Before: Measurement outcome is certain/determ. if $|\psi\rangle$ is eigenvector. BUT: No two Paulis have joint eigenvector.

$\{X, Z\} := XZ + ZX = 0$

↳ uncertainty in either X or Z measurement (or both)

How to quantify? Consider

$p_X(x)$ = prob. of outcome x when meas. X

$$|\langle\psi|X|\psi\rangle| = |p_X(1) - p_X(-1)| = |2p_X(1) - 1| \leq 1$$

* $=1$ if outcome certain,
  $=0$ if outcome completely random

$\ast$ Uncertainty principle: For every state $|\psi\rangle$,

$$\boxed{|\langle\psi|X|\psi\rangle| + |\langle\psi|Z|\psi\rangle| \leq \sqrt{2} < 2}$$

Proof: Let $s_x, s_z \in \{\pm 1\}$. Then:

$$s_x\langle\psi|X|\psi\rangle + s_z\langle\psi|Z|\psi\rangle$$

$$= \langle\psi|\underbrace{\boxed{s_x X + s_z Z}}_{=:\,A}|\psi\rangle \underset{CS}{\leq} \|A\| := \sup_{\|\phi\|=1} \|A\phi\|$$

operator norm

Note:

$$A^\dagger A = A^2 = I + s_x s_z \underbrace{(XZ + ZX)}_{=0} + I = 2 \cdot I$$

$$\Rightarrow \|A\|^2 = \|A^\dagger A\| = 2 \qquad \square$$

NB: Works for every state (just like it should)!

↳ PSet for more discussion.