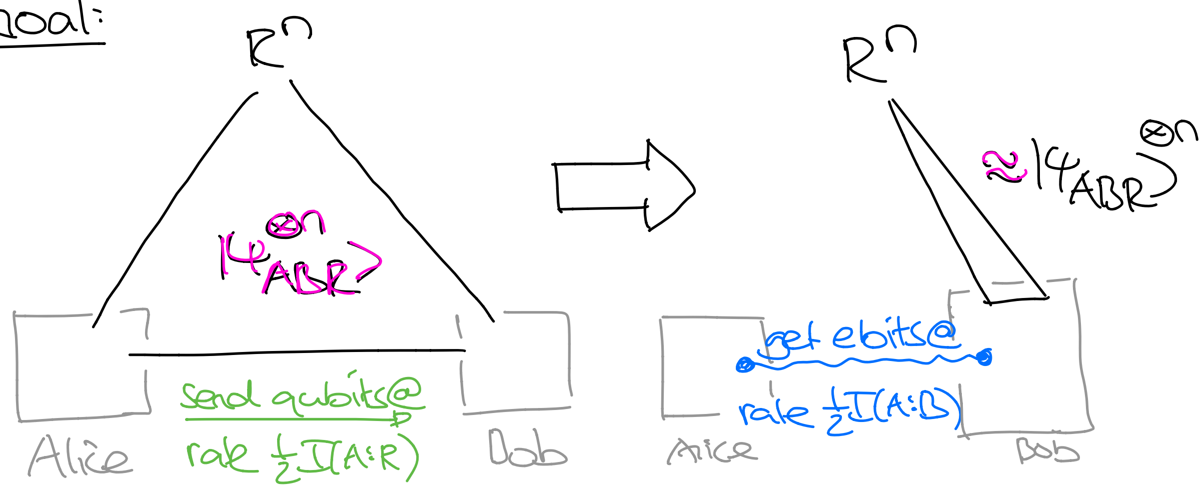
Yesterday: Entropy & mutual information

Quantum State merging

a.k.a. Coherent q state merging, fully q. Slepian-Wolf, ...

Goal:



* For Comparison: q. state transfer needs qubit rate $S(A) \geq \frac{1}{2}I(A:B)$, yields no ebits
* Other variants possible: q. state splitting, redistribution, ...

Applications: i.e. $B = \mathbb{C}$

* No B: Q. state transfer $\to$ $\frac{1}{2}I(A:R) = S(A)$, optimal!
* Entanglement distillation:

  $\boxed{\rho_{AB}^{\otimes n} \Rightarrow \approx |\Phi^+\rangle^{\otimes R \cdot n} \text{ by sending classical bits}}$

  Send qubits by teleportation:

  send bits at rate: $I(A:R)$

  get ebits at rate: $\frac{1}{2}I(A:B) - \frac{1}{2}I(A:R) = \underbrace{S(B) - S(AB)}_{\text{Coherent info}}$

  Can be negative

No R: Obtain ebits at rate $S(B) = S(A)$! without communication

* <u>Noisy teleportation</u>: Using $\rho_{AB}^{\otimes n}$:

First distill ebits, then do ordinary teleportation!

Send bits at rate $I(A:R) + 2\big(S(B) - S(AB)\big) = I(A:B)$

↳ teleport qubits at rate $S(B) - S(AB)$ ← if ≥ 0

* <u>Noisy superdense coding</u>: Using $\rho_{AB}^{\otimes n}$:

Send qubits at rate $\frac{1}{2} I(A:R) + \frac{1}{2} I(A:B) = S(A)$

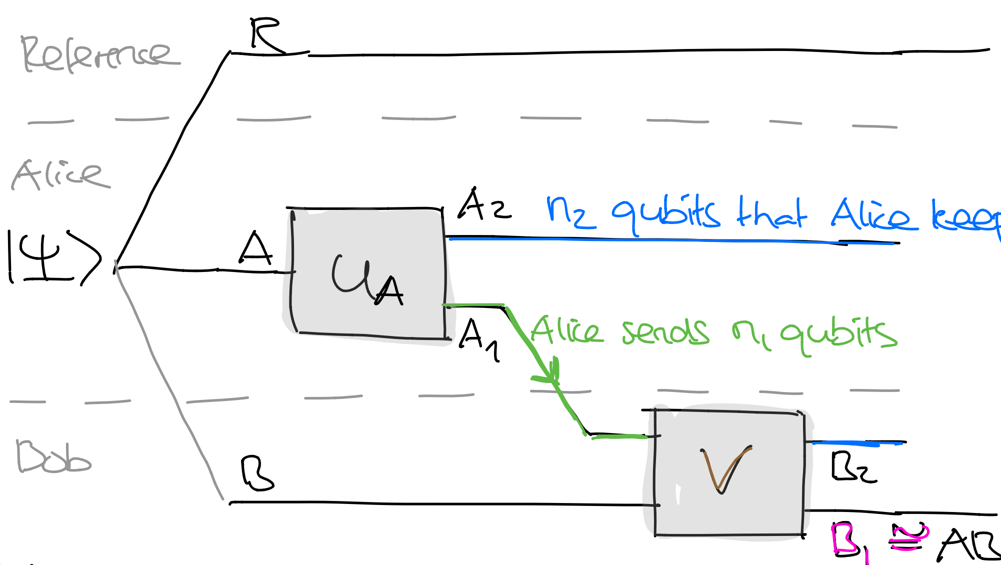
↳ Communic. bits at rate $I(A:B)$

only interesting if ">" i.e. if $S(B) - S(AB) > 0$

$|\psi_{ABR}\rangle = |\Phi^+_{AB}\rangle$: ordinary teleportation + superdense coding

How to solve? Consider first $|\psi\rangle^{\otimes n} \rightsquigarrow |\psi\rangle$?

<u>Want</u>: Unitary U_A & isometry V s.th. circuit



yields state

$$\boxed{\approx \underbrace{|\Phi^+\rangle^{\otimes n_2}}_{\text{on } A_2B_2} \otimes |\Psi\rangle_{B_1R}.} \quad (*)$$

(and hopefully n_1 not too large = n_2 not too small)

Decoupling approach: Enough to consider

$$|\Gamma\rangle_{ABR} = (U_A \otimes I_B \otimes I_R)|\Psi_{ABR}\rangle$$

* Necessary & sufficient:

$$\boxed{\Gamma_{A_2R} \approx \frac{I_{A_2}}{2^{n_2}} \otimes \Psi_R}$$

☺ Get V "for free"

"decoupled" from environment

Why sufficient? (*) is purification!

* Decoupling theorem: Let Ψ_{AR} on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_R}$, $d_A = d_{A_1} \cdot d_{A_2}$. Then:

$$\boxed{\int dU_A \left\| \overbrace{\mathrm{tr}_{A_1}\left[U_A \Psi_{AR} U_A^\dagger\right]}^{\Gamma_{A_2R}} - \frac{I_{A_2}}{d_{A_2}} \otimes \Psi_R \right\|_1^2 \leq \frac{d_A d_R}{d_{A_1}^2} \mathrm{tr}\left[\Psi_{AR}^2\right]}$$

Small if we trace out enough...

* How to use? Apply with typical projectors for ψ_A, ψ_B, ψ_R

$$|\Psi\rangle = (\boxed{P_{A,n}} \otimes \boxed{P_{B,n}} \otimes \boxed{P_{R,n}}) |\psi_{ABR}\rangle^{\otimes n} \underset{\text{AEP}}{\approx} |\psi_{ABR}^{\otimes n}\rangle$$

$$\overset{!}{\in} \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_R} \subseteq$$

$$\Rightarrow d_A d_R \underbrace{\mathrm{tr}[\Psi_{AR}^2]}_{= \mathrm{tr}[\Psi_B^2]} \underset{\text{AEP}}{\leq} 2^{n(S(A)+\varepsilon + S(R)+\varepsilon - S(AR)+\varepsilon)}$$

<u>RESULT</u>: Decoupling succeeds if

$$\underbrace{\frac{1}{n}\log(d_{A_1})}_{\text{qubit rate}} \approx \frac{1}{2} I(A:R)$$

$$\Rightarrow \underbrace{\frac{1}{n}\log(d_{A_2})}_{\text{ebit rate}} \approx S(A) - \frac{1}{2}I(A:R) = \frac{1}{2}I(A:B)$$

AEP, have $d_A \geq 2^{n(S(A)-\varepsilon)}$ from eigenvalue bound!

We still need to prove the decoupling theorem...

Haar averages:

Compare $\boxed{\text{PSET 3}}$ $(d=2)$

1. For all X on $\mathbb{C}^d$: irrep

$$\underbrace{\int dU\, U X U^\dagger}_{U(d)\text{-intertwiner}} = \frac{\mathrm{tr}[X]}{d} \cdot I$$

since $\mathrm{Sym}^2 + \Lambda^2$ irreps

2. For all X on $\mathbb{C}^d \otimes \mathbb{C}^d$:

$$\int dU\, U^{\otimes 2} X (U^\dagger)^{\otimes 2} = \overbrace{\alpha \cdot I + \beta \cdot F}^{= \gamma \cdot \Pi_2 + \delta (I - \Pi_2)}$$

Here:

$$\alpha = \frac{d\,\mathrm{tr}[X] - \mathrm{tr}[FX]}{d^3 - d} \quad \& \quad \beta = \frac{d\,\mathrm{tr}[FX] - \mathrm{tr}[X]}{d^3 - d}$$

In general: $\int dU\, U^{\otimes n} \ldots U^{\dagger \otimes n} \in \mathrm{span}\{R_\pi\}$

Sanity check:

$$\int dU\, \mathrm{tr}_{A_1}[U_A \Psi_{AR} U_A^\dagger] \overset{1}{=} \int dU\, \mathrm{tr}_{A_1}\left[\frac{I_A}{d_A} \otimes \Psi_R\right]$$

$$= \frac{I_{A_2}}{d_{A_2}} \otimes \Psi_R$$

...so there is hope...

Proof of decoupling theorem:

we skipped this in class

* Use $\|M\|_2 := \sqrt{\mathrm{tr}[M^\dagger M]} = \sqrt{\sum s_i^2}$. Note:

$$\underset{\text{for us: } d_{A_2} d_R}{\|M\|_1 \leq \sqrt{d} \cdot \|M\|_2} \qquad \underbrace{\sum_i s_i \leq \sqrt{\sum 1} \cdot \sqrt{\sum s_i^2}}_{\text{Cauchy-Schwartz}}$$

* $$\left\| \mathrm{tr}_{A_1}[U_A \Psi_{AR} U_A^\dagger] - \frac{I_{A_2}}{d_{A_2}} \otimes \Psi_R \right\|_2^2$$

$$= \mathrm{tr}\left[\mathrm{tr}_{A_1}[U_A \Psi_{AR} U_A^\dagger]^2\right] - \underbrace{\frac{1}{d_{A_2}} \mathrm{tr}[\Psi_R^2]}_{\text{indep. of } U}$$

average?

$$\int dU \, \mathrm{tr}\left[\mathrm{tr}_{A_1}[U_A \Psi_{AR} U_A^\dagger]^2\right]$$

swap trick

$$= \mathrm{tr}\Big[\Psi_{AR}^{\otimes 2} \underbrace{\int dU \, U_A^{\dagger \otimes 2} (I_{A_1 A_1'} \otimes F_{A_2 A_2'}) U_A^{\otimes 2}}_{= \alpha \cdot I_{AA'} + \beta \cdot F_{AA'}} \otimes F_{RR'}\Big]$$

...calculate...

$$= \ldots \leq \underbrace{\frac{1}{d_{A_2}} \mathrm{tr}[\Psi_R^2]}_{\text{cancels}} + \underbrace{\frac{1}{d_{A_1}} \mathrm{tr}[\Psi_{AR}^2]}_{\text{survives}}$$

Together:

$$\int dU \, \|\ldots\|_1^2 \leq d_{A_2} d_R \int dU \, \|\ldots\|_2^2 \leq \frac{d_{A_2} d_R}{d_{A_1}} \mathrm{tr}[\Psi_{AR}^2] \qquad \square$$

What we did NOT cover:

* Noisy q. communication channels + their capacities
  to send bits, qubits, ...

* Converses, i.e. why the obtained rates were optimal

GOOD LUCK FOR THE EXAM ☺