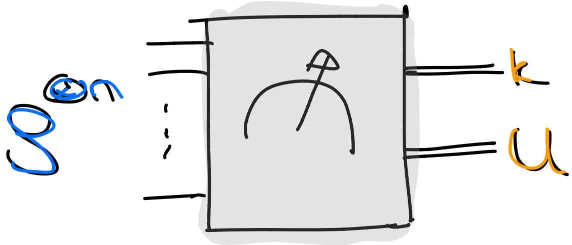
Quantum State Tomography (Contd.)



$$\hat{\varrho} = U \begin{pmatrix} \hat{p} & \\ & 1-\hat{p} \end{pmatrix} U^\dagger \approx \varrho$$

where $\hat{p} = \frac{1}{2}\left(1 + \frac{k}{n}\right)$

↳ POVM:

$$\boxed{Q_{k,U} = \frac{k+1}{\mathrm{tr}\left[T^{(n,k)}_{\hat{\varrho}}\right]} P_{n,k}\, \hat{\varrho}^{\otimes n} P_{n,k}}$$

Analysis: Compute probability density

$$\mathrm{tr}\left[Q_{k,U}\, \varrho^{\otimes n}\right] \leftarrow \text{show small unless } \varrho \approx \hat{\varrho}$$

$$= \frac{k+1}{\mathrm{tr}\left[T^{(n,k)}_{\hat{\varrho}}\right]} \mathrm{tr}\left[P_{n,k}\, \hat{\varrho}^{\otimes n} P_{n,k}\, \varrho^{\otimes n}\right]$$

$$= \frac{k+1}{\mathrm{tr}\left[T^{(n,k)}_{\hat{\varrho}}\right]} \mathrm{tr}\left[T^{(n,k)}_{\hat{\varrho}}\, T^{(n,k)}_{\varrho} \otimes I_{W_{n,k}}\right]$$

$$\leq \frac{(n+1)2^{n h(\hat{p})}}{\mathrm{tr}\left[T^{(n,k)}_{\hat{\varrho}}\right]} \mathrm{tr}\left[T^{(n,k)}_{\underbrace{\boxed{\sqrt{\varrho}\,\hat{\varrho}\sqrt{\varrho}}}_{X^2}}\right]$$

$$X := \sqrt{\sqrt{\varrho}\,\hat{\varrho}\sqrt{\varrho}}$$

$$\mathrm{tr}[X] = F(\varrho, \hat{\varrho})$$

$$\sigma := \frac{X}{\mathrm{tr}[X]}$$

$$= \frac{(n+1)2^{nh(\hat{p})}}{\underbrace{\boxed{\mathrm{tr}[T^{(n,k)}_{\hat{\varrho}}]}}_{①}} \underbrace{\boxed{\mathrm{tr}[T^{(n,k)}_{\sigma^2}]}}_{②} \cdot F(\varrho,\hat{\varrho})^{2n}$$

Let's bound both terms:

$$① \quad \mathrm{tr}[T^{(n,k)}_{\hat{\varrho}}] = (\det \hat{\varrho})^{\frac{n-k}{2}} \underbrace{\mathrm{tr}[T^{(k)}_{\hat{\varrho}}]}_{=\hat{p}^k+\dots} \qquad = \mathrm{tr}[\cancel{T^{(k)}_{(\cdot)}} T^{(k)}_{(\cdot,\cdot)} \cancel{T^{(k)}_{(\cdot)}}]$$

$$\geq \hat{p}^{\frac{n+k}{2}}(1-\hat{p})^{\frac{n-k}{2}} \geq 2^{-nh(\hat{p})}$$

② If σ has eigenvalues $\{q, 1-q\}$:

$$\mathrm{tr}[T^{(n,k)}_{\sigma^2}] = (\det\sigma)^{2\frac{n-k}{2}} \underbrace{\mathrm{tr}[T^{(n,k)}_{\sigma^2}]}_{=q^{2k}+\dots}$$

$$\leq (n+1)\, q^{2\frac{n+k}{2}}(1-q)^{2\frac{n-k}{2}} = (n+1)\,2^{-2(h(\hat{p})+\underbrace{\delta(\hat{p}\|q)}_{\geq 0})}$$

$$\leq (n+1)2^{-2nh(\hat{p})}$$

generalization of bound in spectrum estimation (2 vs. 1)

Together:

all entropy terms cancel!

$$\boxed{\mathrm{tr}[Q_{k,U}\varrho^{\otimes n}] \leq (n+1)^2 F(\varrho,\hat{\varrho})^{2n}}$$

$$\ldots = \sum_k \int du \; \mathbb{1}_{[F(\rho,\hat\rho)\le 1-\delta]} \operatorname{tr}[Q_{k,u}\,\rho^{\otimes n}] \le \ldots$$

RESULT:

$$\Pr_{\rho^{\otimes n}}\left(F(\hat\rho,\rho)\le 1-\delta\right) \le (n+1)^3(1-\delta)^{2n} \longrightarrow 0$$

exponentially

SUCCESS :)

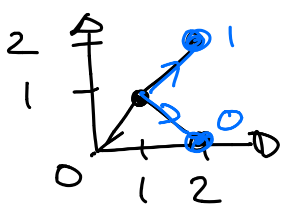
PS: See SCHUR-WEYL TOOLBOX in lecture notes for summary of all we did so far.

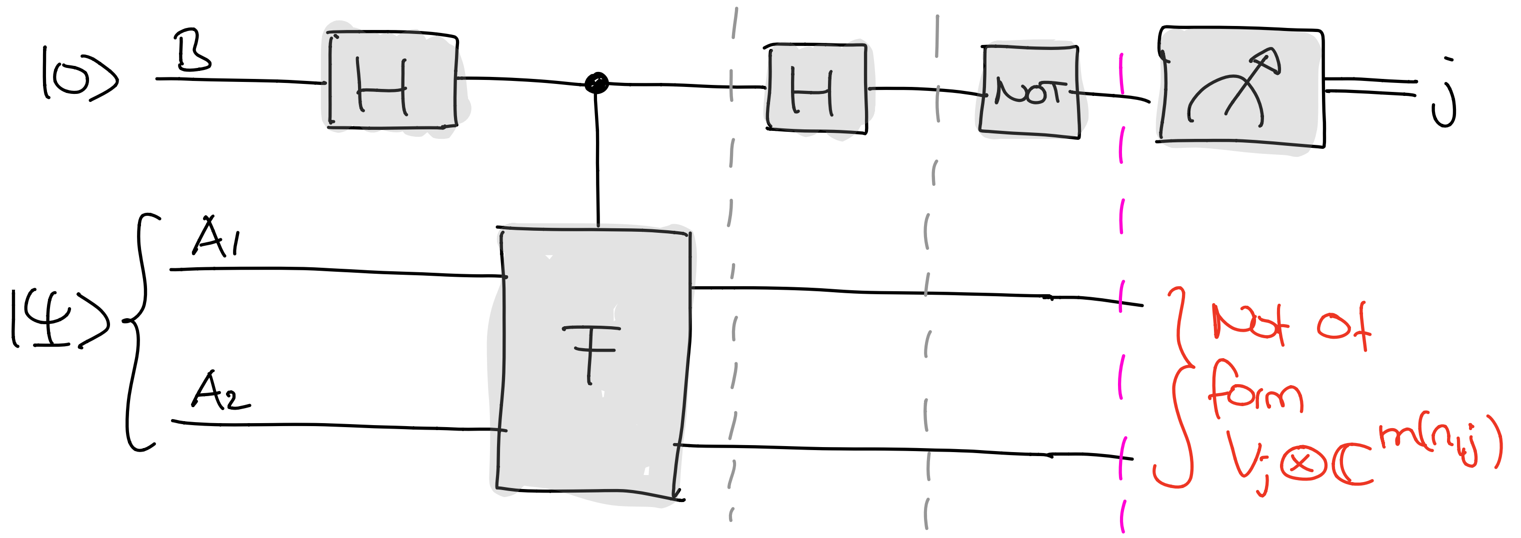
Goal for rest of today: Quantum circuit that implements $(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_k V_{n,k}\otimes W_{n,k}$!

Towards a q. circuit for the Schur transform

Warmup: $\mathbb{C}^2\otimes\mathbb{C}^2$

$$= \underbrace{\mathrm{Sym}^2(\mathbb{C}^2)}_{V_{2,2}} \oplus \underbrace{\mathbb{C}|\Psi^-\rangle}_{V_{2,0}}$$





$$\frac{1}{\sqrt{2}}\left[|0\rangle_B \otimes |\psi\rangle_{A_1A_2} + |1\rangle_B \otimes F|\psi\rangle_{A_1A_2}\right]$$

$$\frac{1}{\sqrt{2}}\left(\frac{\overset{1}{|0\rangle}+\overset{0}{|1\rangle}}{\sqrt{2}} \otimes |\psi\rangle + \frac{\overset{1}{|0\rangle}-\overset{0}{|1\rangle}}{\sqrt{2}} \otimes F|\psi\rangle\right)$$

$$= \frac{1}{2}\left[\overset{1}{|0\rangle} \otimes (|\psi\rangle + F|\psi\rangle) + \overset{0}{|1\rangle} \otimes (|\psi\rangle - F|\psi\rangle)\right]$$

$$= \overset{1}{|0\rangle} \otimes \underbrace{\boxed{\Pi_2|\psi\rangle}}_{\in Sym^2} + \overset{0}{|1\rangle} \otimes \underbrace{\boxed{(I-\Pi_2)|\psi\rangle}}_{\in \mathbb{C}|\psi^-\rangle}$$

Circuit implements: $\boxed{|\psi\rangle \overset{V}{\longmapsto} \sum_{j=0}^{1} |j\rangle \otimes P_{2,2j}|\psi\rangle}$

i.e. $\tilde{\rho} \longmapsto \sum_{j,\tilde{j}} |j\rangle\langle\tilde{j}| \otimes P_j \tilde{\rho} P_{\tilde{j}}$

↳ $\Pr(\text{outcome } j) = \operatorname{tr}[P_{2,2j}\, \tilde{\rho}_{A_1A_2}]$

↳ $\Pr_{\rho\otimes\sigma}(\text{outcome } 1) = \frac{1}{2}\left(1 + \operatorname{tr}[\rho\sigma]\right)$

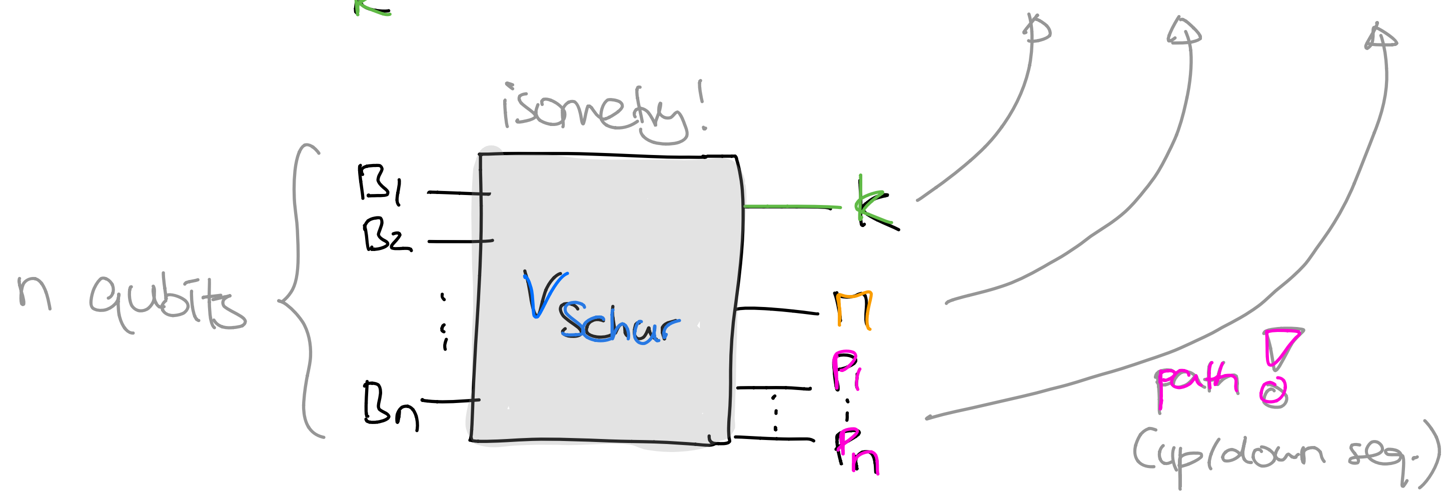
$\rho^{\otimes 2}$: purity

$|\psi\rangle \otimes |\phi\rangle$: $|\langle\psi|\phi\rangle|^2$

A q. circuit for the Schur transform

Suffices to consider SU(2) (like in our derivation)

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_k \mathrm{Sym}^k(\mathbb{C}^2)\otimes \mathbb{C}^{m(n,k)} \hookrightarrow \mathbb{C}^{n+1}\otimes\mathbb{C}^{n+1}\otimes\mathbb{C}^{2^n}$$



* e.g.: spectrum estimation ⟶ measure k register

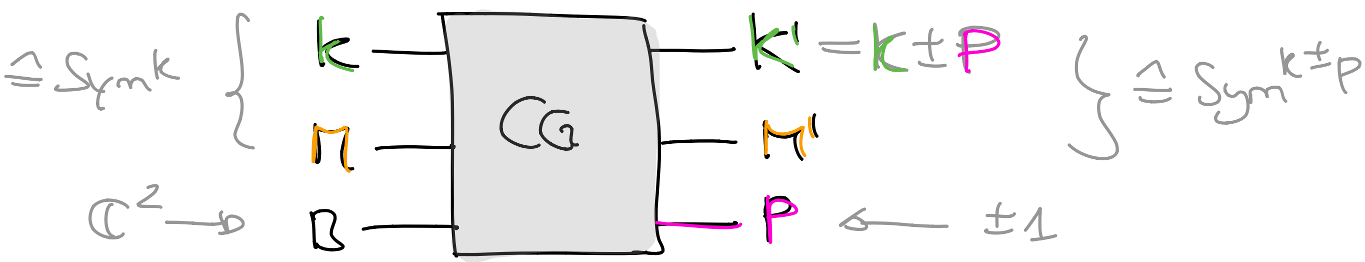
<u>key ingredient:</u>

$$J:\ \mathrm{Sym}^k(\mathbb{C}^2)\otimes\mathbb{C}^2 \cong \bigoplus_{p=\pm 1} \mathrm{Sym}^{k+p}(\mathbb{C}^2)$$

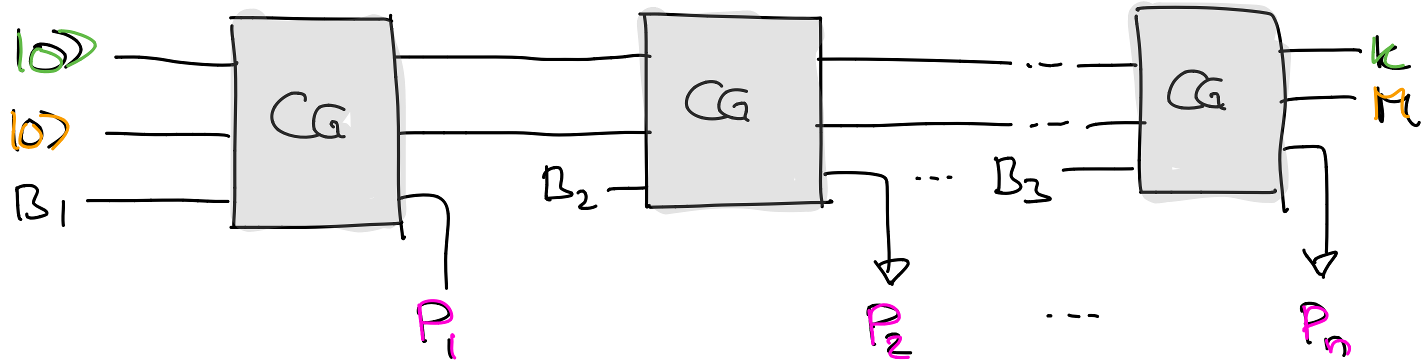
Clebsch-Gordan coeff's (NEXT TIME)

<u>Plan:</u>

① Build a circuit for CG: with variable k



② Obtain Schur transform from:



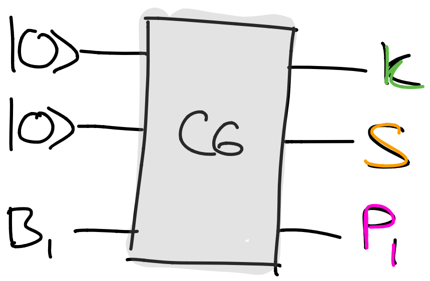
Implements

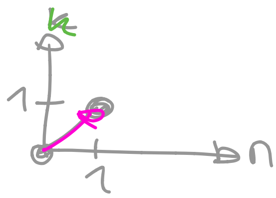
$$|b_1\rangle \cdots |b_n\rangle \in (\mathbb{C}^2)^{\otimes n} \cong \bigoplus_k \overbrace{\boxed{\mathrm{Sym}^k(\mathbb{C}^2)}}^{M} \otimes \overbrace{\boxed{\mathbb{C}^{m(n,k)}}}^{P_1 \cdots P_n}$$

$$\subseteq \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \otimes \overbrace{\boxed{(\mathbb{C}^2)^{\otimes n}}}^{\text{path}}$$

Examples:

n=1:

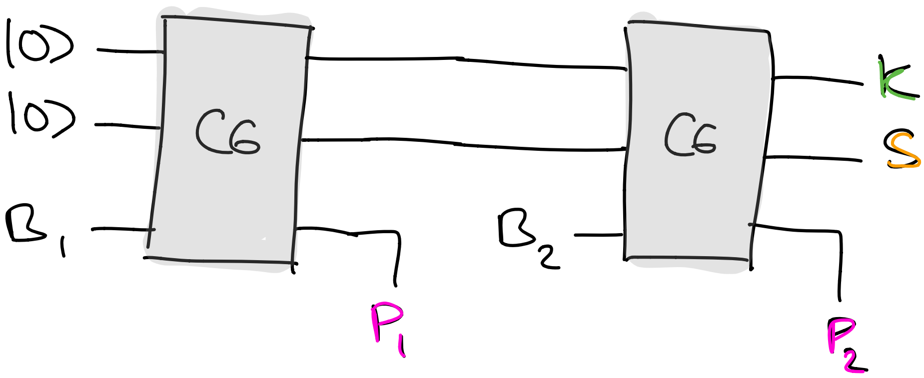


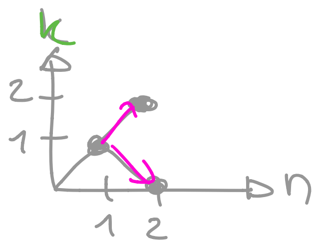


$$|0\rangle \mapsto |1\rangle_k \otimes |{+1}\rangle_S \otimes |+\rangle_P$$

$$|1\rangle \mapsto |1\rangle_k \otimes |{-1}\rangle_S \otimes |+\rangle_P$$

n=2:





$$|0\rangle_{B_1}|0\rangle_{B_2} \longmapsto |2\rangle_k \otimes |2\rangle_S \otimes |++\rangle_P$$

$$|0\rangle_{B_1}|1\rangle_{B_2} \longmapsto \sqrt{\tfrac{1}{2}}\,|2\rangle_k \otimes |0\rangle_S \otimes |++\rangle_P$$

$$|1\rangle_{B_1}\ |0\rangle_{B_2} \qquad \pm\sqrt{\tfrac{1}{2}}\,|0\rangle_k \otimes |0\rangle_S \otimes |+-\rangle_P$$

$$|1\rangle_{B_1}|1\rangle_{B_2} \longmapsto |2\rangle_k \otimes |-2\rangle_S \otimes |++\rangle_P$$

$$\left(\begin{rcases} |0\rangle|1\rangle \\ |1\rangle|0\rangle \end{rcases} = \sqrt{\tfrac{1}{2}}\ \frac{|01\rangle+|10\rangle}{\sqrt{2}} \pm \sqrt{\tfrac{1}{2}}\ \frac{|01\rangle-|10\rangle}{\sqrt{2}} \right)$$