Last week: Spec. estimation & Schur-Weyl duality

Odds & ends: We proved for $m(\lambda) \equiv n(\lambda)$ → Same proof here

* Lemma: Any intertwiner $J: \bigoplus_\lambda V_\lambda \otimes \mathbb{C}^{m(\lambda)} \to \bigoplus_\mu V_\mu \otimes \mathbb{C}^{n(\mu)}$ is of form $\bigoplus_\lambda I_{V_\lambda} \otimes J_\lambda$ where $J_\lambda: \mathbb{C}^{m(\lambda)} \to \mathbb{C}^{n(\lambda)}$

* E.g. $V_\lambda \to \bigoplus_\mu V_\mu \otimes \mathbb{C}^{n(\mu)}$ is of form "$I_\lambda \otimes |\psi\rangle$" ($|\psi\rangle \in \mathbb{C}^{n(\lambda)}$, i.e. $\bigoplus_\mu \delta_{\lambda\mu} I_\lambda \otimes |\psi\rangle$)

useful for PSET 6 !

* Lemma: $\forall M: [M, R_\pi] = 0 \ \forall \pi \in S_n \Rightarrow M \in \operatorname{span}\{X^{\otimes n}\}$

Proof sketch: We know that (looks similar)

$$\forall |\Phi\rangle \in \operatorname{Sym}^n(\mathbb{C}^d) \Rightarrow |\Phi\rangle \in \operatorname{span}\{|\psi\rangle^{\otimes n}\} \qquad (\ast)$$

Now use iso $M \mapsto |\Phi\rangle := (M \otimes I) \sum_{\vec{x}} |\vec{x}, \vec{x}\rangle$

($R_\pi \quad R_\pi^T \quad (R_\pi \otimes R_\pi)$)

See PSET 5 for lifting symmetries

* QMath Master Class on Tensors @ Copenhagen: June 18-22

Deadline: March 30

GOAL TODAY: Q. State tomography! $\rho^{\otimes n} \to \hat{\rho} \approx \rho$

Fidelity

Distance measures so far:

- Trace dist: $T(\rho,\sigma) = \max_{0\leq Q\leq I} \left| tr[Q\rho] - tr[Q\sigma] \right|$
- Fidelity for pure states: $|\langle\psi|\phi\rangle| = \sqrt{1 - T^2(\psi,\phi)}$

How about mixed states?

Define $F(\rho_A, \sigma_A) = \sup_{\substack{R,\, |\Psi_{AR}\rangle,\, |\Phi_{AR}\rangle \\ \text{purifications}}} |\langle\Psi_{AR}|\Phi_{AR}\rangle|$

* For pure states: $F(\psi_A, \phi_A) = |\langle\psi_A|\phi_A\rangle|$ ☺

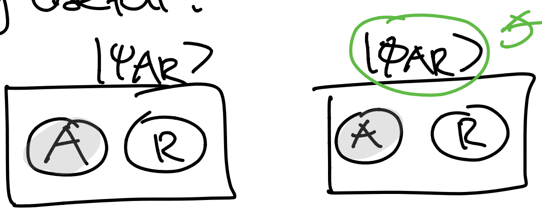
$\rho_A = |\psi\rangle\langle\psi_A| \rightsquigarrow |\Psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ etc.

* Monotonicity: $F(\rho_A, \sigma_A) \geq F(\rho_{AB}, \sigma_{AB})$ cf. PSET

$|\Psi_{ABC}\rangle$ purification of $\rho_{AB} \to$ also of ρ_A

* If $\exists$ purifications $|\Psi_{AR}\rangle, |\Phi_{AR}\rangle$: This R suffices! (& sup is max!)

* Why useful?



← favorite purification

$\rho_A \approx \sigma_A$

$\Rightarrow |\psi_{AR}\rangle \approx (I \otimes U_R)|\phi_{AR}\rangle$

* Fuchs-van de Graaf inequalities:

$$1-F \le T \le \sqrt{1-F^2}$$

"=" if both states pure

Harder, see a book. :(

choose $|\Psi_{AR}\rangle, |\Phi_{AR}\rangle$ that achieve F

$\Rightarrow T(\rho_A, \sigma_A) \le T(\Psi_{AR}, \Phi_{AR}) = \sqrt{1-F^2}$.

PSETS

* Concrete formula: $F(\rho, \sigma) = \operatorname{tr}\sqrt{\sqrt{\sigma}\,\rho\,\sqrt{\sigma}} = \|\sqrt{\rho}\sqrt{\sigma}\|_1$

any purification of ρ is of form $(I \otimes V)(\sqrt{\rho} \otimes I)\sum_i |ii\rangle$

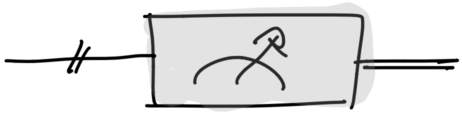
partial isometry

trace norm for general M

$$\|M\|_1 := \operatorname{tr}\sqrt{M^\dagger M}$$

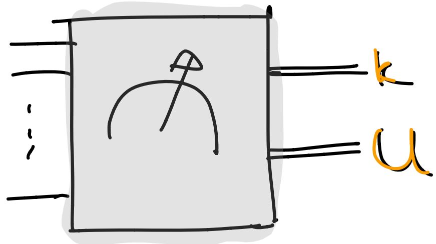
$$= \sum \text{s.v. of } M$$

Quantum State Tomography

$\rho^{\otimes n}$  $\hat{\rho} \approx \rho$? (Fidelity)

We know how to estimate spectrum & also when $\rho = |\psi\rangle\langle\psi|$. Combine!

Idea: Design POVM $\{Q_{~~\lambda~~, k, U}\}$ s.th.

$\rho^{\otimes n}$ 

$$\hat{\rho} = U \begin{pmatrix} \hat{p} & \\ & 1-\hat{p} \end{pmatrix} U^{\dagger} \approx \rho$$

where $\hat{p} = \frac{1}{2}\left(1 + \frac{k}{n}\right)$

* POVM must satisfy:

$$Q_{k,U} \geq 0 \quad \& \quad \sum_k \int dU \, Q_{k,U} = I$$

↑ Haar measure: unique prob. measure s.th. $\int dU \, f(U) = \int dU \, f(VUW) \quad \forall V, W$

* Want that coarse-grains to spectrum estimation:

$$\int dU \, Q_{k,U} \overset{!}{=} P_{n,k} \quad (\forall k) \qquad (*)$$

ANSATZ:

$$Q_{k,U} \propto P_{n,k} \, \hat{\rho}^{\otimes n} \, P_{n,k} \cong T_{\hat{\rho}}^{(n,k)} \otimes I_{W_{n,k}}$$

What if we measure on pure states?

* $k=n$: $P_{n,n} = \Pi_n$ $\quad \hat{\rho} = U \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} U^\dagger = |\psi\rangle\langle\psi|$

↳ $Q_{n,n} \propto |\psi\rangle\langle\psi|^{\otimes n}$ uniform POVM on Sym^n ☺

* Symmetries: $[R_\pi, Q_{k,U}]$ $\forall \pi, k, U$

& $\mathrm{tr}[\rho^{\otimes n} Q_{k,U}]$

$= \mathrm{tr}[(V\rho V^\dagger)^{\otimes n} Q_{k,VU}]$ Covariant ☺

$$\begin{array}{ccc} \rho & \longrightarrow & \hat{\rho} \\ \downarrow & & \downarrow \\ V\rho V^\dagger & \longrightarrow & V\hat{\rho}V^\dagger \end{array}$$

* How about (*)? $\underbrace{\int dU\, T^{(n,k)}_{\hat{\rho}}}_{\text{Intertwiner, hence } \propto I_{V_{n,k}}} \otimes I_{W_{n,k}}$

$$\int dU\, T^{(n,k)}_{U(\hat{p},\, 1-\hat{p})U^\dagger} \overset{\text{Subst. } U \text{ by } VU}{=} \int dU\, T^{(n,k)}_{VU(\hat{p},\, 1-\hat{p})U^\dagger V} \quad \& \quad T^{(n,k)}_{XY} = T^{(n,k)}_X T^{(n,k)}_Y$$

$$\Rightarrow \int dU\, \underbrace{P_{n,k}\, \hat{\rho}^{\otimes n} P_{n,k}}_{} \propto \underbrace{P_{n,k}}_{} \quad ☺$$

$\mathrm{tr} = \underbrace{\mathrm{tr}[T^{(n,k)}_{\hat{\rho}}]}_{\text{does not depend on } U!} \cdot m(n,k)$ $\qquad \mathrm{tr} = (k+1)\, m(n,k)$

↳ POVM:

$$Q_{k,U} = \frac{k+1}{\mathrm{tr}[T^{(n,k)}_{\hat{\rho}}]} P_{n,k}\, \hat{\rho}^{\otimes n} P_{n,k}$$

Analysis: Compute

$$\operatorname{tr}\left[Q_{k,u}\, \rho^{\otimes n}\right] \leftarrow$$ will show that small unless $\rho \approx \hat\rho$

(probability density)

$$= \frac{k+1}{\operatorname{tr}\left[T_{\hat\rho}^{(n,k)}\right]} \operatorname{tr}\left[P_{n,k}\, \hat\rho^{\otimes n} P_{n,k}\, \rho^{\otimes n}\right]$$

$$= \frac{k+1}{\operatorname{tr}\left[T_{\hat\rho}^{(n,k)}\right]} \operatorname{tr}\left[T_{\hat\rho}^{(n,k)} T_{\rho}^{(n,k)} \otimes I_{W_{n,k}}\right]$$

$$\leq \frac{(n+1)2^{n h(\hat p)}}{\operatorname{tr}\left[T_{\hat\rho}^{(n,k)}\right]} \operatorname{tr}\left[T^{(n,k)}_{\sqrt{\rho}\hat\rho\sqrt{\rho}}\right]$$

$\operatorname{tr}\sqrt{\sqrt{\rho}\,\hat\rho\,\sqrt{\rho}} = F(\rho,\hat\rho)$!

$$\leq (n+1)^2 \boxed{F(\rho,\hat\rho)^{2n}}$$

☺

Will prove this tomorrow by lower- & upper-bounding $\operatorname{tr}[T^{(n,k)}_{\dots}]$!