Yesterday: <u>Compression</u>

* Classical data source: $p_0 = p_H = p$, $p_1 = p_T = 1-p$

<u>Typical sequence:</u> $b = b_1 \cdots b_n \in \{0,1\}^n$ with $\left| \frac{\#0\text{'s}}{n} - p \right| \leq \varepsilon$ ← parameter

$\#$ of typical sequences $\leq (n+1) \cdot 2^{n(h(p)+\varepsilon')}$

* Quantum data source: $\rho = \sum_x p_x |\psi_x\rangle\langle\psi_x|$ average output of source

To compress at rate R, want <u>typical subspaces</u>

$\mathcal{H}_n \subseteq (\mathbb{C}^2)^{\otimes n}$ with projector P_n such that

① $\operatorname{tr}[P_n \rho^{\otimes n}] \longrightarrow 1$ ← typical

② $\frac{\log(\dim \mathcal{H}_n)}{n} \leq R$ ← size $\leq R \cdot n$ qubits

<u>How to construct?</u> Take spectral decomposition of ρ

$$\rho = p \cdot |\phi_0\rangle\langle\phi_0| + (1-p) \cdot |\phi_1\rangle\langle\phi_1|$$

orthogonal!

and define NOT ψ_x!

$$\mathcal{H}_n = \operatorname{Span}\{ |\phi_{b_1}\rangle \otimes \cdots \otimes |\phi_{b_n}\rangle : \vec{b} \in \{0,1\}^n \text{ typical sequence} \}$$

Then:

① $\operatorname{tr}[\rho^{\otimes n} P_n] = \sum_{\vec{b} \text{ typical}} \underbrace{\langle \phi_{b_1} \otimes \cdots \otimes \phi_{b_n} | \rho^{\otimes n} | \phi_{b_1} \otimes \cdots \otimes \phi_{b_n} \rangle}_{= p^{\#0\text{'s}} (1-p)^{\#1\text{'s}}}$

$= \Pr(\vec{b}$ typical for <u>classical</u> data source with $p_0 = p$, $p_1 = 1-p)$

$\longrightarrow 1$ (Law of large numbers)

② dim($\mathcal{H}_n$) = # typical sequences

$$\Rightarrow \frac{\log(\dim \mathcal{H}_n)}{n} \leq \underbrace{\frac{\log(n+1)}{n}}_{\to 0} + \boxed{h(p)} + \underbrace{\varepsilon'}_{\text{arbitrarily small}}$$

RESULT: Can compress at asymptotic qubit rate arbitrarily close to

$$h(p) =: S(\rho) = -\mathrm{tr}[\rho \cdot \log \rho]$$ von Neumann entropy

* rate is optimal
* protocol works for all q. sources described by ρ!
* Symmetries? $S(\rho) = S(U\rho U^\dagger)$ — we use the eigenbasis!

:(Our protocol for ρ does not work for $U\rho U^\dagger$

! Despite ρ appearing everywhere... goal of compression is NOT to prepare Bob in state ρ. !

Next lectures: Will use representation theory to study $\rho^{\otimes n}$ and solve these problems (+ other) problems!

universal compression

spectrum estimation

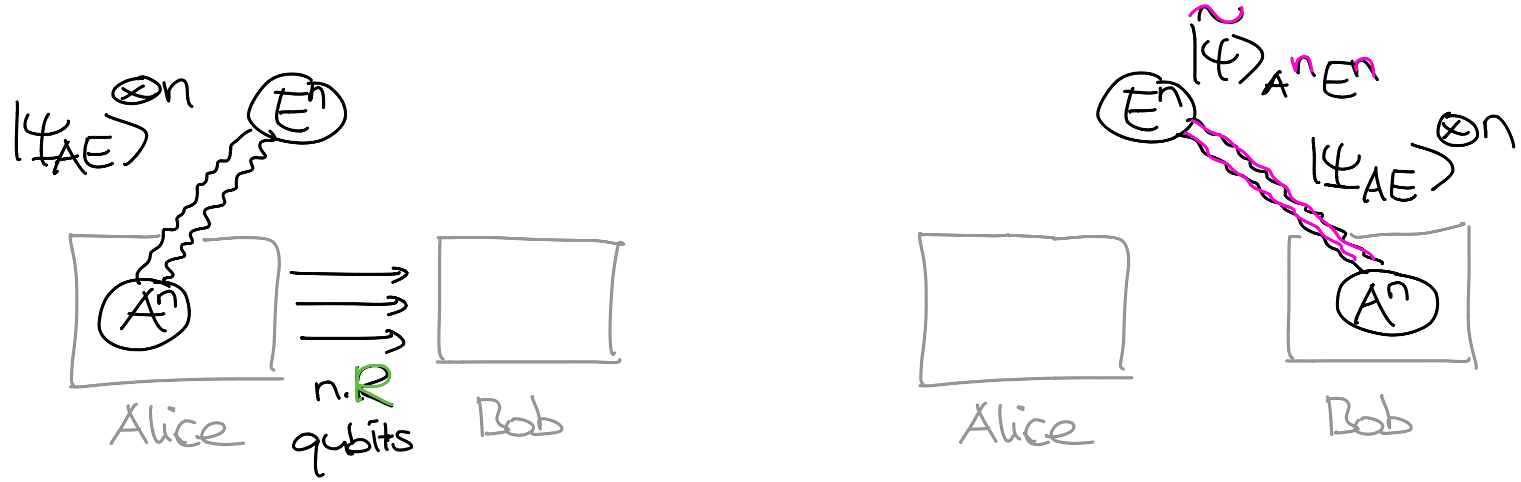
entropy estim.

(MIXED STATE) TOMOGRAPHY

[Q.] Compression is about minimizing [q.] communication...

Rest of today: <u>Compression & entanglement</u> — another such scenario

<u>Q. State transfer:</u> Given state $|\Psi_{AE}\rangle^{\otimes n}$, Alice wants to transfer A-systems over to Bob at min. qubit rate:



* Both Alice + Bob know state $|\Psi_{AE}\rangle$
* E need NOT be in Alice' laboratory
* Intuition: The more <u>entangled</u>, the more qubits:

$|\Psi_{AE}\rangle = |\Psi_A\rangle \otimes |\Phi_E\rangle \rightsquigarrow$ Bob can prepare $|\Psi_B\rangle$ $\boxed{R=0}$
no correlation — No communication necessary!

<u>Idea:</u> Use typical subspaces $\mathcal{H}_{A,n} \subseteq (\mathbb{C}^2)^{\otimes n}$ for $\rho_A = \mathrm{tr}_E[|\Psi_{AE}\rangle\langle\Psi_{AE}|]$

projectors

<u>Protocol:</u> Alice measures $\{P_{A,n}, I - P_{A,n}\}$

typical — atypical → FAIL

Post-measurement state

$$|\tilde{\Psi}_{A^nE^n}\rangle = \frac{(P_{A,n}\otimes I_{E^n})|\Psi_{AE}^{\otimes n}\rangle}{\|(P_{A,n}\otimes I_{E^n})|\Psi_{AE}^{\otimes n}\rangle\|} \in \boxed{\mathcal{H}_{A,n}} \otimes \mathcal{H}_E^{\otimes n}$$

2

Alice sends over A-systems: $\approx n\cdot(S(\rho_A)+\varepsilon')$ qubits

Analysis: * $Pr(\text{SUCCESS}) = \langle \Psi_{AE}^{\otimes n} | P_{A,n}\otimes I_{E^n} | \Psi_{AE}^{\otimes n}\rangle$

$$= \underbrace{\mathrm{tr}[\rho_A^{\otimes n} P_{A,n}]}_{=:q} \xrightarrow{1} 1$$

* ...and in this case:

$$|\langle\tilde{\Psi}_{A^nE^n}|\Psi_{AE}^{\otimes n}\rangle|^2 = \frac{q^2}{q} = \mathrm{tr}[\rho_A^{\otimes n} P_{A,n}] \xrightarrow{1} 1$$

SUMMARY: Can transfer at asymptotic qubit rate (arbitrarily close to)

2nd operational interpret. of v.N. entropy

$$S(\rho_A) =: S(A)_\Psi \overset{\text{Schmidt}}{\underset{\text{decomps.}}{=}} S(B)_\Psi =: S_E(\Psi_{AB})$$

Notation

Entanglement entropy of pure state $|\Psi_{AB}\rangle$

* e.g. $|\Psi_{AE}\rangle = |0\rangle\otimes|0\rangle \longrightarrow S_E = 0$

$|\Psi_{AE}\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}} \longrightarrow S_E = 1$

NOTATION:

- E is for entanglement
- has nothing to do with E system

* any protocol for state transfer can be used to compress quantum source with $\rho = \rho_A$.

* Interesting variant: If Bob already has part of system ...

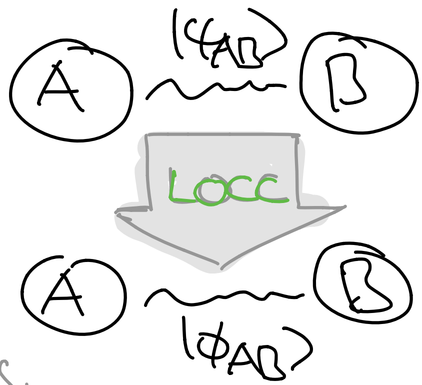
Outlook: Entanglement Transformations

Pure state entanglement: $|\psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\psi\rangle_B$

How to compare? How to quantify?

Study possible transformations:

$$|\psi_{AB}\rangle \xrightarrow[\text{LOCC}]{} |\phi_{AB}\rangle$$



- Local Operations   unitaries, measurements, add $\rho_{\tilde{A}}$, $\mathrm{tr}_{\tilde{A}}$, ...
- **Classical** Communication   e.g. meas. results

UNLIKE ABOVE IN COMPRESSION !!!

CANNOT CREATE ENT. FROM NOTHING

"$|\psi_{AB}\rangle$ is at least as useful as $|\phi_{AB}\rangle$"

Idea: Use ebit $|\Phi^+\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$ as "currency".

... Study (asymptotic) conversion rates from/to ebits:

Entanglement cost: Study

$$|\Phi^+\rangle^{\otimes Rn} \xrightarrow{\text{LOCC}} |\tilde{\Psi}_{A^nB^n}\rangle \approx |\Psi_{AB}\rangle^{\otimes n}$$

$E_C(\Psi) :=$ "min. asympt. rate" R $\left(\inf_{\varepsilon>0} \limsup_{n\to\infty} R\right)$

Distillable entanglement: Study

$$|\Psi_{AB}\rangle^{\otimes n} \xrightarrow{\text{LOCC}} |\tilde{\Psi}_{A^nB^n}\rangle \approx |\Phi^+\rangle^{\otimes Rn}$$

$E_D(\Psi) :=$ "max. asymptotic rate" R

FACT: For pure states: $\boxed{E_C = E_D = S_E}$

FUTURE PSET?