The formalism of quantum information theory

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This handout summarizes the formalism of quantum information theory that we have developed in this course, starting from the axioms of quantum mechanics.

(A) **Systems**: To every quantum mechanical system, we associate a *Hilbert space* $\mathcal{H}$. For a joint system composed of two subsystems A and B, with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the Hilbert space is the tensor product $\mathcal{H}_{AB} \coloneqq \mathcal{H}_A \otimes \mathcal{H}_B$.

(B) **States**: A *density operator* ρ is an operator on $\mathcal{H}$ that satisfies (i) $\rho \geq 0$ and (ii) $\mathrm{tr}[\rho] = 1$. Any density operator describes the state of a quantum mechanical system. If the rank of ρ is one (i.e., of the form $\rho = \psi \coloneqq |\psi\rangle\langle\psi|$ for some unit vector $|\psi\rangle \in \mathcal{H}$) then we say that ρ is a *pure state*. Otherwise, ρ is called a *mixed state*. An *ensemble* $\{p_i, \rho_i\}$ of quantum states can be described by the density operator $\rho = \sum_i p_i \rho_i$.

If ρ_{AB} is the state of a joint system, the state of its subsystems can be described by the *reduced density matrices* $\rho_A = \mathrm{tr}_B[\rho_{AB}]$ and $\rho_B = \mathrm{tr}_A[\rho_{AB}]$. The latter states can be mixed even if ρ_{AB} is pure. Conversely, any density operator ρ_A has a *purification* $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ (see Lectures 7 and 8).

(C) **Unitary dynamics**: Given a *unitary* operator U on $\mathcal{H}$, the transformation $\rho \mapsto U\rho U^\dagger$ is in principle physical. In other words, the laws of quantum mechanics allow a way of evolving the quantum system for some finite time such that, when we start in an arbitrary initial state ρ, the final state is $U\rho U^\dagger$. If $\rho = |\psi\rangle\langle\psi|$ is a pure state, then this corresponds to $|\psi\rangle \mapsto U|\psi\rangle$.

(D) **Measurements**: A *POVM measurement* $\{Q_x\}_{x\in\Omega}$ with outcomes in some finite set Ω is a collection of operators on $\mathcal{H}$ that satisfies (i) $Q_x \geq 0$ and (ii) $\sum_{x\in\Omega} Q_x = I$. Born's rule asserts that the probability of outcome x in state ρ is given by the *Born rule*:

$$\Pr_\rho(\text{outcome } x) = \mathrm{tr}\left[\rho Q_x\right].$$

If $\rho = |\psi\rangle\langle\psi|$ is a pure state, then this can also be written as $\langle\psi|Q_x|\psi\rangle$. A POVM measurement that has precisely two outcomes is called a *binary POVM measurement*, and it has the form $\{Q, I - Q\}$, hence is specified by a single POVM element $0 \leq Q \leq I$. We can also consider POVMs with a continuum of possible outcomes (see Lecture 4).

We say that $\{P_x\}$ is a *projective measurement* if $\{P_x\}_{x\in\Omega}$ is a POVM where the P_x are projections that are pairwise orthogonal (i.e., $Q_x Q_y = \delta_{x,y} Q_x$). If $\Omega \subseteq \mathbb{R}$, then the data $\{P_x\}_{x\in\Omega}$ is equivalent to specifying a Hermitian operator with spectral decomposition $O = \sum_x x P_x$, called an *observable*. If the outcome of a projective measurement is x then the state of the system "collapses" into the *post-measurement state*

$$\rho' = \frac{P_x \rho P_x}{\mathrm{tr}[P_x \rho]}$$

If $\rho = |\psi\rangle\langle\psi|$ is a pure state, then $\rho' = |\psi'\rangle\langle\psi'|$, where $|\psi'\rangle = P_x|\psi\rangle / \|P_x|\psi\rangle\|$.

Any POVM can be implemented using projective measurements on a larger system (see Lecture 2).

(E) **Operations on subsystems:** Consider a joint system with Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. If we want to perform a unitary U_A on the subsystem modeled by $\mathcal{H}_A$, then the appropriate unitary on the joint system is $U_A \otimes I_B$. Similarly, if $\{Q_{A,x}\}_{x\in\Omega}$ is a POVM measurement on $\mathcal{H}_A$ then the appropriate POVM measurement on the joint system is $\{Q_{A,x} \otimes I_B\}_{x\in\Omega}$.

The standard formalism of quantum information theory includes two further notions that we did not discuss in this course: *Quantum channels* model general evolutions that can be obtained by composing unitary dynamics, adding ancillas, and taking partial traces. *Quantum instruments* Can be thought of as implementations of POVM measurements that not only describe the statistics of outcomes but also model the post-measurement state.