PHYSICS 491: Symmetry and Quantum Information	May 2, 2017
Problem Set 4	
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Problem 1 (Schur-Weyl duality).
Your goal in this exercise is to concretely identify irreducible representations of $U(2)$ and S_n in the n-qubit Hilbert space. Let j be such that $\frac{n}{2} - j$ is a nonnegative integer.

(a) Show that the subspace

$$\mathcal{H}_{n,j} := \left\{ |\phi\rangle \otimes |\psi^-\rangle^{\otimes \frac{n}{2}-j},\ |\phi\rangle \in \operatorname{Sym}^{2j}(\mathbb{C}^2) \right\} \subseteq (\mathbb{C}^2)^{\otimes n}$$

is an irreducible $U(2)$-representation equivalent to $V_{n,j}$. Here, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ is the singlet state. How can you obtain further $U(2)$-representations in $(\mathbb{C}^2)^{\otimes n}$ equivalent to $V_{n,j}$?

(b) Now construct an irreducible S_n-representation in $(\mathbb{C}^2)^{\otimes n}$ that is equivalent to $W_{n,j}$. How can you obtain further S_n-representations in $(\mathbb{C}^2)^{\otimes n}$ equivalent to $W_{n,j}$?

(c) Using part (b), confirm that the definition of $W_{\square\square\square}$ and $W_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$ via Schur-Weyl duality is equivalent to our original definition in lecture 3.

Problem 2 (PPT criterion).
In this exercise, you will study a simple, highly useful entanglement criterion. Given an operator M_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, we define its *partial transpose* as the operator $M_{AB}^{T_B}$ with matrix elements

$$\langle a, b | M_{AB}^{T_B} | a', b' \rangle = \langle a, b' | M_{AB} | a', b \rangle .$$

Note that this definition depends on the choice of basis for $\mathcal{H}_B$ (but not of the basis for $\mathcal{H}_A$).

(a) Show that $\operatorname{tr} M_{AB}^{T_B} = \operatorname{tr} M_{AB}$.

(b) Observe that if $M_{AB} = X_A \otimes Y_B$ then $M_{AB}^{T_B} = X_A \otimes Y_B^T$ and argue that this uniquely determines the partial transpose.

In particular, we can consider the partial transpose of a density operator ρ_{AB}.

(c) Show that if ρ_{AB} is separable then $\rho_{AB}^{T_B} \geq 0$.

You thus obtain the so-called *PPT criterion*, short for positive partial transpose criterion: *If the partial transpose* $\rho_{AB}^{T_B}$ *is* not *positive semidefinite then* ρ_{AB} *must be entangled.*

(d) Verify using the PPT criterion that the ebit $|\Psi_2^+\rangle$ is entangled.

(e) Consider the family of *isotropic two-qubit states*,

$$\rho_{AB}(p) := p\,\tau_{\text{sym}} + (1-p)\tau_{\text{anti}},$$

where τ_{sym} denotes the maximally mixed state on the symmetric subspace of two qubits and $\tau_{\text{anti}} = |\psi^-\rangle\langle\psi^-|$ the singlet state. For which values of $p \in [0,1]$ does the PPT criterion establish entanglement?

In general, the PPT criterion is only a sufficient, but not a necessary criterion for entanglement. If $\dim H_A \otimes H_B > 6$, then there exist entangled states with a positive semidefinite partial transpose.

Problem 3 (Dual representations)**.**
This problem introduces the concept of a *dual representation*. To start, consider a representation $\mathcal{H}$ of some group G, with operators $\{R_g\}$. Let $\mathcal{H}^*$ denote the dual Hilbert space, whose elements are "bras" $\langle\phi|$, and define operators R_g^* on $\mathcal{H}^*$ by $R_g^* \langle\phi| := \langle\phi| R_{g^{-1}}$.

(a) Verify that the operators $\{R_g^*\}$ turn $\mathcal{H}^*$ into a representation of G. This representation is called the *dual representation* of $\mathcal{H}$.

(b) Show that if $\mathcal{H}$ is irreducible then $\mathcal{H}^*$ is irreducible.

A representation $\mathcal{H}$ is called *self-dual* if $\mathcal{H}^* \cong \mathcal{H}$.

(c) Show that the irreducible representations of $\mathrm{SU}(2)$, and hence all its representations, are self-dual.

(d) Show that any representation of S_3 is self-dual.

It is true more generally that any representation of S_n is self-dual.

Problem 4 (Many copies of a bipartite pure state)**.**
In this exercise, we will revisit the universal entanglement concentration protocol discussed in lecture 8. Let $|\phi\rangle_{AB}$ be an arbitrary state of two qubits. Then $|\phi\rangle_{AB}^{\otimes n}$ is a vector in the Hilbert space

$$(\mathbb{C}^2)^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n} \cong \left(\bigoplus_j V_{n,j}^A \otimes W_{n,j}^A\right) \otimes \left(\bigoplus_{j'} V_{n,j'}^B \otimes W_{n,j'}^B\right) \cong \bigoplus_{j,j'} V_{n,j}^A \otimes V_{n,j'}^B \otimes W_{n,j}^A \otimes W_{n,j'}^B.$$

The superscripts A refer to the Schur-Weyl decomposition of the n A-systems, and likewise for B. Now consider the representation of S_n on $W_{n,j}^A \otimes W_{n,j'}^B$ given by the operators $R_\pi^{(n,j)} \otimes R_\pi^{(n,j')}$. A vector in $W_{n,j}^A \otimes W_{n,j'}^B$ is called an *invariant vector* if it is left unchanged by all these operators.

(a) Show that if $j \neq j'$ then $W_{n,j}^A \otimes W_{n,j'}^B$ contains no nonzero invariant vector for S_n.

(b) Show that $W_{n,j}^A \otimes W_{n,j}^B$ contains a unique invariant vector (up to scalar multiples). Moreover, show that this vector is a maximally entangled state, which we denote by $|\Phi^+\rangle_{W_{n,j}^A W_{n,j}^B}$.

Hint: Use problem 3 and Schur's lemma.

(c) Conclude that $|\psi\rangle_{AB}^{\otimes n}$ can be written in the form

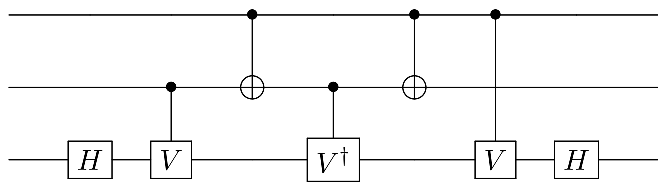
$$|\psi\rangle_{AB}^{\otimes n} \cong \sum_j \sqrt{p_j}\, |\Psi\rangle_{V_{n,j}^A V_{n,j}^B} \otimes |\Phi^+\rangle_{W_{n,j}^A W_{n,j}^B},$$

where $p_j = \mathrm{tr}[P_j \rho_A^{\otimes n}]$ and where the $|\Psi\rangle_{V_{n,j}^A V_{n,j}^B}$ are suitable pure states in $V_{n,j}^A \otimes V_{n,j}^B$.

(d) Use part (c) to analyze the universal entanglement concentration protocol discussed in class.

Problem 5 (The controlled swap gate)**.**
In this exercise, you will decompose the controlled swap (CSWAP) gate into a quantum circuit that consists of single-qubit and two-qubit gates only.

(a) Compute the three-qubit unitary that corresponds to the following quantum circuit:



Here, $V = \begin{pmatrix} 1 & \\ & i \end{pmatrix}$ is a square root of the Z-gate.

The unitary from part (a) is known as the *Toffoli gate*.

(b) Show that the controlled swap gate can be implemented by a sequence of Toffoli gates.