PHYSICS 491: Symmetry and Quantum Information	April 23, 2017
Problem Set 3	
Michael Walter, Stanford University	*due May 4, 2017*

Problem 1 (The antisymmetric state)**.**
In class, we discussed the quantum de Finetti theorem for the symmetric subspace. It asserts that the reduced density matrices $\rho_{A_1\dots A_k}$ of a state on $\mathrm{Sym}^n(\mathbb{C}^d)$ are $\sqrt{kd/(n-k)}$ close in trace distance to a separable state (in fact, to a mixture of tensor power states).

The goal of this exercise is to show that a dependence on the dimension d is unavoidable. To start, consider the *Slater determinant*

$$|S\rangle_{A_1\dots A_d} = |1\rangle \wedge \dots \wedge |d\rangle := \sqrt{\frac{1}{d!}} \sum_{\pi \in S_d} \mathrm{sign}(\pi)\, |\pi(1)\rangle \otimes \ldots \otimes |\pi(d)\rangle \in (\mathbb{C}^d)^{\otimes d}.$$

We define the *antisymmetric state* on $\mathbb{C}^d \otimes \mathbb{C}^d$ by tracing out all but two subsystems,

$$\rho_{A_1 A_2} = \mathrm{tr}_{A_3\dots A_d}\left[|S\rangle\langle S|\right].$$

(a) Show that $T(\rho_{A_1A_2}, \sigma_{A_1A_2}) \geq \frac{1}{2}$ for all separable states $\sigma_{A_1A_2}$.

Hint: Consider the POVM element $Q = \Pi_2$ *(i.e., the projector onto the symmetric subspace).*

Thus you have shown that the antisymmetric state is far from any separable state. However, note that $|S\rangle$ is *not* in the symmetric subspace.

(b) Show that $|S\rangle^{\otimes 2} \in \mathrm{Sym}^d(\mathbb{C}^d \otimes \mathbb{C}^d)$, while $\rho^{\otimes 2}$ is likewise far away from any separable state. Conclude that the quantum de Finetti theorem must have a dependence on the dimension d.

Problem 2 (De Finetti and mean field theory)**.**
In this exercise you will explore the consequences of the quantum de Finetti theorem for mean field theory. Consider an operator h on $\mathbb{C}^d \otimes \mathbb{C}^d$ and the corresponding *mean-field Hamiltonian*

$$H = \frac{1}{n-1} \sum_{i \neq j} h_{i,j}$$

on $(\mathbb{C}^d)^{\otimes n}$, where each term $h_{i,j}$ acts by the operator h on subsystems i and j and by the identity operator on the remaining subsystems (e.g., $h_{1,2} = h \otimes \mathbb{1}^{\otimes(n-2)}$).

(a) Show that the eigenspaces of H are invariant subspaces for the action of the symmetric group.

Now assume that the ground space is nondegenerate, and spanned by some $|E_0\rangle$. Then part (a) implies that $R_\pi |E_0\rangle = \chi(\pi)|E_0\rangle$ for some function χ. This function necessarily satisfies $\chi(\pi\tau) = \chi(\pi)\chi(\tau)$.

(b) Show that $\chi(i \leftrightarrow j) = \chi(1 \leftrightarrow 2)$ for all $i \neq j$. Conclude that $|E_0\rangle$ is either a symmetric tensor or an antisymmetric tensor.

Hint: First show that $\chi(\pi\tau\pi^{-1}) = \chi(\tau)$.

If $n > d$, then there exist no nonzero antisymmetric tensors. Thus, in the thermodynamic limit of large n, the ground state $|E_0\rangle$ is in the symmetric subspace $\mathrm{Sym}^n(\mathbb{C}^d)$ and so the quantum de Finetti theorem is applicable.

(c) Show that, for large n, the energy density in the ground state can be well approximated by

$$\frac{E_0}{n} \approx \min_{|\psi\rangle} \langle \psi^{\otimes 2}|h|\psi^{\otimes 2}\rangle = \frac{1}{n} \min_{|\psi\rangle} \langle \psi^{\otimes n}|H|\psi^{\otimes n}\rangle \,.$$

This justifies the folklore that "in the mean field limit the ground state has the form $|\psi\rangle^{\otimes\infty}$".

Problem 3 (Universal quantum data compression).
In class, we discussed a quantum compression protocol that works for all qubit ensembles $\{p_x, |\psi_x\rangle\}$ for which the associated density operator $\rho = \sum_x p_x |\psi_x\rangle\langle\psi_x|$ has given eigenvalues $\{p, 1-p\}$.

Your task in this exercise is to design a *universal compression protocol* that works for all qubit ensembles with $S(\rho) < S_0$, where $S_0 > 0$ is a given target compression rate.

(a) Show that, for all $S_0 > 0$, there exist projectors $\tilde{P}_n$ on subspaces $\tilde{\mathcal{H}}_n$ of $(\mathbb{C}^2)^{\otimes n}$ such that:

  (i) For all density operators ρ with $S(\rho) < S_0$, $\mathrm{tr}\left[\tilde{P}_n \rho^{\otimes n}\right] \to 1$ as $n \to \infty$,

  (ii) The dimension of $\tilde{\mathcal{H}}_n$ is at most $2^{n(S_0+\delta(n))}$ for some function δ with $\delta(n) \to 0$ as $n \to \infty$.

  Hint: Use the spectrum estimation projectors P_j in a clever way.

(b) Use the projectors $\tilde{P}_n$ to construct a compression protocol with compression rate S_0 that works for all qubit ensembles with $S(\rho) < S_0$ (i.e., show that in the limit of large block length n, the average squared overlap between the original state and the decompressed state goes to one).

  Hint: Follow the same construction as in lecture 7.

Bonus Problem 4 (Bounds on entropies).
In this exercise, you will prove two bounds that we used in class. Let $0 \le p, q \le 1$. The first bound concerns the binary entropy function $h(p) = -p\log p - (1-p)\log(1-p)$.

(a) Consider the function $\eta(x) = -x\log x$ and assume that $|p-q| \le \frac{1}{2}$. Show that

$$|\eta(p) - \eta(q)| \le \eta(|p-q|), \tag{3.1}$$

  and deduce the following special case of *Fannes' inequality*:

$$|h(p) - h(q)| \le 2\eta(|p-q|)$$

The second bound concerns the binary relative entropy $\delta(p\|q) = p\log\frac{p}{q} + (1-p)\log\frac{1-p}{1-q}$.

(b) Derive the following special case of *Pinsker's inequality*:

$$\delta(p\|q) \ge \frac{2}{\ln 2}(p-q)^2.$$

Hint: Remember that $\log x = \ln x/\ln 2$ *is the logarithm to the base two.*

Bonus Problem 5 (Schur-Weyl duality).
In class, we discussed an important mathematical result known as Schur-Weyl duality. The goal of this exercise is to supply some last details and conclude its proof.

Recall that we decomposed the Hilbert space of n qubits as a representation of $U(2)$. Using the same notation as in class,

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j V_{n,j} \otimes \mathbb{C}^{m(n,j)},$$

such that, for all $X \in U(2)$,

$$X^{\otimes n} \cong \bigoplus_j T_X^{(n,j)} \otimes \mathbb{1}_{\mathbb{C}^{m(n,j)}}, \tag{3.2}$$

and we discussed that this formula can be extended to arbitrary operators X on $\mathbb{C}^2$.

(a) Show that the representation operators R_π for $\pi \in S_n$ have the form

$$R_\pi \cong \bigoplus_j \mathbb{1}_{V_{n,j}} \otimes R_\pi^{(n,j)}. \tag{3.3}$$

Conclude that the operators $R_\pi^{(n,j)}$ turn the spaces $\mathbb{C}^{m(n,j)}$ into representations of S_n. We will denote these representations by $W_{n,j}$.

Hint: Recall that $[U^{\otimes n}, R_\pi] = 0$ *and use Schur's lemma.*

In view of eqs. (3.2) and (3.3), we observe that $[X^{\otimes n}, R_\pi] = 0$ for *arbitrary* operators X on $\mathbb{C}^2$.

(b) Show that, conversely, any operator that commutes with all R_π can be written as a linear combination of operators of the form $X^{\otimes n}$.

Hint: Compute $\frac{d}{dt_1}\Big|_{t_1=0} \cdots \frac{d}{dt_n}\Big|_{t_n=0} \left(\sum_{i=1}^n t_i X_i\right)^{\otimes n}$. *Why does this help?*

(c) Conclude that the representations $W_{n,j}$ of S_n are irreducible and pairwise inequivalent.

Hint: Use Schur's lemma.

You have thus proved the following result, known as *Schur-Weyl duality*: The decomposition

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j V_{n,j} \otimes W_{n,j}$$

holds as a representation of both $U(2)$ and S_n. The spaces $V_{n,j}$ and $W_{n,j}$ are pairwise inequivalent, irreducible representations of $U(2)$ and of S_n, respectively. This has important consequences. E.g.:

(d) Show that any operator that commutes with all $U^{\otimes n}$ and R_π is necessarily of the form $\sum_j z_j P_j$, with $z_j \in \mathbb{C}$. Conclude that $\{P_j\}$ is the most fine-grained projective measurement that has both symmetries of the spectrum estimation problem, as discussed in class.

Hint: Use Schur's lemma.