PHYSICS 491: Symmetry and Quantum Information	April 16, 2017
Problem Set 2	
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Problem 1 (Pure state entanglement).
In this exercise you will study the entanglement of pure states $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$. In class, we discussed the Schmidt decomposition

$$|\psi\rangle_{AB} = \sum_{i=1}^{r} s_i \, |e_i\rangle_A \otimes |f_i\rangle_B$$

and its relation to the eigenvalues of the reduced density matrices. For simplicity we will assume that $\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$.

(a) We say that $|\psi\rangle_{AB}$ is *maximally entangled* if $s_i = \frac{1}{\sqrt{d}}$ for all i. Show that $|\psi\rangle_{AB}$ is maximally entangled if and only if ρ_A and ρ_B are maximally mixed (i.e., proportional to $\mathbb{1}$).

(b) Show that $|\psi\rangle_{AB}$ is a product state if and only if ρ_A and ρ_B are pure states.

This suggests that the eigenvalues of the reduced density matrices ρ_A and ρ_B can be used to characterize the entanglement of $|\psi\rangle_{AB}$. As an example, consider the *Rényi-2 entropy*, defined by

$$S_2(A) = -\log \operatorname{tr} \rho_A^2.$$

(c) Find a formula for $S_2(A)$ in terms of the eigenvalues of the reduced density matrices.

(d) Show that $S_2(A) = 0$ for product states, $S_2(A) = \log d$ for maximally entangled states, and otherwise $0 < S_2(A) < \log d$.

You will now study the average entanglement of pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$, drawn at random from the "uniform" probability distribution $d\psi_{AB}$ that you know from class.

(e) Let F_A denote the swap operator on $\mathcal{H}_A^{\otimes 2}$ that sends $|a_1, a_2\rangle \mapsto |a_2, a_1\rangle$. Verify that

$$\operatorname{tr} \rho_A^2 = \operatorname{tr}\left[(F_A \otimes \mathbb{1}_{BB}) \, |\psi\rangle_{AB}^{\otimes 2} \langle\psi|_{AB}^{\otimes 2}\right].$$

(f) Let F_B denote the swap operator on $\mathcal{H}_B^{\otimes 2}$, defined in the same way as F_A. Show that

$$\int d\psi_{AB} \, |\psi\rangle_{AB}^{\otimes 2} \langle\psi|_{AB}^{\otimes 2} = \frac{1}{d^2(d^2+1)} \left(\mathbb{1}_{AA} \otimes \mathbb{1}_{BB} + F_A \otimes F_B\right).$$

Hint: Remember the symmetric subspace.

(g) Show that the average Rényi-2 entropy $S_2(A)$ of a random pure state is no smaller than $\log d - \log 2$.

Hint: Jensen's inequality shows that $\int d\psi \, \log f(|\psi\rangle) \leq \log\left(\int d\psi \, f(|\psi\rangle)\right)$.

Problem 2 (Extensions of quantum states).
In this exercise you will verify two important facts that we discussed in class:

(a) Show that any density operator admits a *purification*. That is, given a quantum state ρ_A on some Hilbert space $\mathcal{H}_A$, construct a pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_B$ is some auxiliary Hilbert space, such that

$$\rho_A = \operatorname{tr}_B \left[|\psi\rangle \langle\psi|_{AB} \right].$$

Hint: Consider the spectral decomposition of ρ_A.

(b) Show that any extension of a pure state is a tensor product. That is, show that if ρ_A is pure then any extension is of the form

$$\rho_{AB} = \rho_A \otimes \rho_B.$$

Hint: You have already solved this problem in the case that ρ_{AB} is pure.

Problem 3 (The symmetric subspace is irreducible).
In this problem, you will show that the symmetric subspace is an irreducible representation of $\mathrm{SU}(d)$. We will start with $d = 2$. For any operator M on $\mathbb{C}^2$, define a corresponding operator on $(\mathbb{C}^2)^{\otimes n}$ by

$$\widetilde{M} = M_1 + M_2 + \dots + M_n.$$

Here we write $M_1 = M \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$, $M_2 = \mathbb{1} \otimes M \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$, etc. Now consider an arbitrary subspace $\mathcal{H} \subseteq \mathrm{Sym}^n(\mathbb{C}^2)$ that is invariant for $\mathrm{SU}(2)$.

(a) Show that $\widetilde{M} |\psi\rangle \in \mathcal{H}$ for any vector $|\psi\rangle \in \mathcal{H}$.

Hint: If H is Hermitian then e^{iH} is unitary.

In class, we observed that the symmetric subspace has natural occupation number basis. For $d = 2$, it is given by

$$\|t\rangle\!\rangle \propto |\underbrace{0, \ldots, 0}_{t}, \underbrace{1, \ldots, 1}_{n-t}\rangle + \text{permutations} \qquad (t = 0, \ldots, n).$$

(b) Find an operator M such that $\widetilde{M}$ has the basis vectors $\|t\rangle\!\rangle$ as eigenvectors (with distinct eigenvalues). Conclude that $\mathcal{H}$ is spanned by a subset of the basis vectors $\|t\rangle\!\rangle$.

(c) Find operators $M_\pm$ such that $\widetilde{M_\pm} \|t\rangle\!\rangle \propto \|t \pm 1\rangle\!\rangle$. Conclude that $\mathcal{H}$ is either $\{0\}$ or all of $\mathrm{Sym}^n(\mathbb{C}^d)$.

Thus you have proved that $\mathrm{Sym}^n(\mathbb{C}^2)$ is indeed an irreducible representation of $\mathrm{SU}(2)$!

(d) Any irreducible representation of $\mathrm{SU}(2)$ can be labeled by its spin j. What is the spin of the symmetric subspace $\mathrm{Sym}^n(\mathbb{C}^2)$?

(e) *Optional:* Sketch how your proof can be generalized to show that $\mathrm{Sym}^n(\mathbb{C}^d)$ is an irreducible representation of $\mathrm{SU}(d)$.

Bonus Problem 4 (Entanglement witnesses and convexity).
An observable X_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called an *entanglement witness* for a quantum state ρ_{AB} if

$$\mathrm{tr}[X_{AB}\,\rho_{AB}] < 0,$$

while

$$\mathrm{tr}[X_{AB}\,\sigma_{AB}] \geq 0 \tag{2.1}$$

for all separable states σ_{AB}.

(a) Construct an entanglement witness for the maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Hint: Compute the overlap of $|\Phi^+\rangle$ *with a pure product state* $|\psi\rangle_A \otimes |\phi\rangle_B$. *Why could this help?*

(b) Argue that for any entangled state ρ_{AB} there exists an entanglement witness X_{AB}.

Hint: You do not need to construct the entanglement witness explicitly.

Bonus Problem 5 (The extendibility hierarchy).
In this problem, you will show that any quantum state that has an n-extension is close to a separable state if n is large, as discussed in class.

(a) Imitate the proof of the quantum de Finetti theorem given in class to show that, for any pure state $|\Phi\rangle_{AB_1\ldots B_n} \in \mathcal{H}_A \otimes \mathrm{Sym}^n(\mathcal{H}_B)$,

$$\mathrm{tr}_{B_2\ldots B_n}[|\Phi\rangle\langle\Phi|] \approx \int d\psi\, p(\psi)\; |W_\psi\rangle\langle W_\psi|_A \otimes |\psi\rangle\langle\psi|_{B_1}$$

for large n. Here, the integral is over the set of pure states on $\mathcal{H}_B$, $p(\psi)$ is a probability density, and the $|W_\psi\rangle$ are pure states in $\mathcal{H}_A$.

Now suppose that ρ_{AB} is an arbitrary quantum state that has an n-extension (i.e., that there exists some $\sigma_{AB_1\ldots B_n}$ such that $\sigma_{AB_k} = \rho_{AB}$ for all k).

(b) Show that ρ_{AB} also has an n-extension $\rho_{AB_1\ldots B_n}$ that is permutation-invariant on the B-systems, i.e., $[\mathbb{1}_A \otimes R_\pi, \rho] = 0$ for all $\pi \in S_n$.

Any n-extension as in (b) admits a purification in $(\mathcal{H}_A \otimes \mathcal{H}_{A'}) \otimes \mathrm{Sym}^n(\mathcal{H}_B \otimes \mathcal{H}_{B'})$, where $\mathcal{H}_{A'} = \mathcal{H}_A$ and $\mathcal{H}_{B'} = \mathcal{H}_B$.

(c) Conclude that any n-extendible ρ_{AB} is close to a separable state for large n.

Hint: The trace distance does not increase when you take the partial trace.