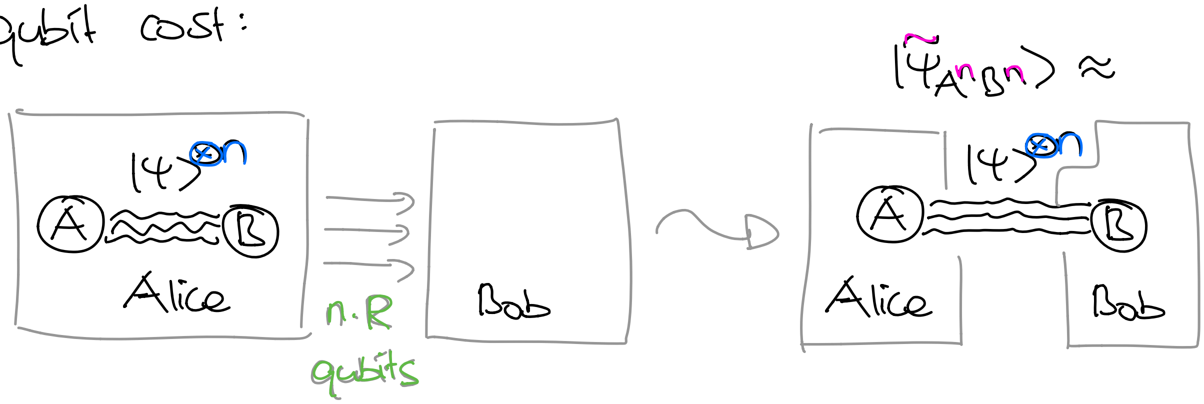
Compression & Entanglement

density ops also appear as reduced density ops

$$\rho_B = |\psi_{AB}\rangle$$

Goal: Send over B-system to Bob at minimal qubit cost:



"quantum state transfer"

Intuition: The more entangled — the more qubits we need to send. E.g.

$|\psi_{AB}\rangle = |0\rangle \otimes |0\rangle \rightsquigarrow$ prepare $|0\rangle^{\otimes n}$ at Bob's side

Protocol: Measure $\tilde{P}_{B^n}$ on B-systems

Outcome 1: Send over B-systems of post-meas. state

Outcome 0: Fail.

$$|\tilde{\psi}_{A^nB^n}\rangle = \frac{(I_{A^n} \otimes \tilde{P}_{B^n})|\psi_{AB}\rangle^{\otimes n}}{\| \dots \|}$$

$$\in (\mathbb{C}^2)^{\otimes n} \otimes \boxed{\tilde{\mathcal{H}}_n}$$

$n \cdot (S(\rho) + \delta)$ qubits by (2)

Analysis:

$$\Pr(\text{outcome } 1) = \langle \psi_{AB}^{\otimes n} | I_{A^n} \otimes \tilde{P}_{B^n} | \psi_{AB}^{\otimes n} \rangle$$

$$= \operatorname{tr}[\rho_B^{\otimes n} \tilde{P}_{B^n}] =: q \overset{\text{by (1)}}{\approx} 1$$

In this case:

$$|\langle \psi_{AB}^{\otimes n} | \tilde{\psi}_{A^nB^n} \rangle|^2 = q \approx 1$$

SUCCESS!

- optimal rate: entanglement entropy

$$S_E = S(\rho_B) = S(\rho_A)$$ 2nd operational interpret.

- e.g. $|\psi_{AB}\rangle = |0\rangle \otimes |0\rangle \longrightarrow S_E = 0$ $\boxed{\text{pset 2}}$

  $|\psi_{AB}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \longrightarrow S_E = 1$ mixedness vs. entanglement

- any protocol for state transfer can be used to compress quantum sources! $\boxed{\text{Pset}}$

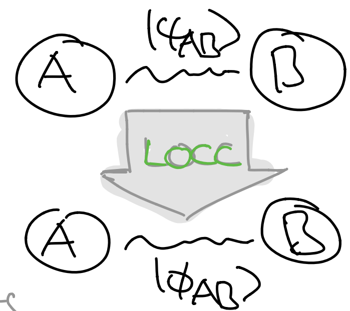
Entanglement transformations

Pure state entanglement: $|\psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\psi\rangle_B$

How to compare? How to quantify?

$S(\rho_A)$, $S_2(\rho_A)$, eigenvalues of ρ_A, ... ?!

Study possible transformations:



$$|\psi_{AB}\rangle \xrightarrow[\text{LOCC}]{} |\phi_{AB}\rangle$$

- Local Operations — unitaries, measurements, add $\rho_{\tilde{A}}$, $\mathrm{tr}_{\tilde{A}}$, ... ← CANNOT CREATE ENT.
- $\boxed{\text{Classical}}$ Communication — e.g. meas. results

  UNLIKE IN COMPRESSION

"$|\psi_{AB}\rangle$ is at least as useful as $|\phi_{AB}\rangle$"

Ex: $|\Phi_2^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ "EBIT", EPR pair, ...

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$$ max. entangled states in d dimensions

Fact: $|\Phi_d^+\rangle \xrightarrow[\text{LOCC}]{} |\Phi_{d'}^+\rangle$ iff $d \geq d'$ $\boxed{\text{Pset}}$

⌈ Obvious if $d = 2^n$, $d' = 2^{n'}$: simply trace out $n-n'$ qubits.

"Only if" using "Schmidt rank". ⌋

General solution: entire "ent. spectrum" matters (NIELSEN)
BUT: Asymptotic theory simplifies tremendously.

Key idea: Use $|\Phi_2^+\rangle$ as "currency" & count.

Entanglement concentration

$$|\psi_{AB}\rangle^{\otimes n} \xrightarrow{\text{LOCC}} |\tilde{\psi}\rangle \approx |\Phi_2^+\rangle^{\otimes Rn}$$

Optimal rate: distillable entanglement E_D

How can we solve this problem?

Idea: First focus on Alice' Hilbert space $\qquad \lambda \leftrightarrow (n,j)$

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j V_j^A \otimes W_j^A$$

$\boxed{\text{SUPRESS "n"}}$

$$\rho_A^{\otimes n} \cong \bigoplus_j T_{\rho_A}^{(n,j)} \otimes I_{W_j^A}$$

rewrite using density operators

$$= \bigoplus_j p_j \, \rho_{V_j^A} \otimes \tau_{W_j^A}$$

$p_j = \Pr(\text{outcome } j)$ $\qquad$ density operators:

e.g. $\tau_{W_j^A} = \dfrac{I_{W_j^A}}{m(n,j)}$ $\qquad$ max. mixed

- Alice measures $\{P_j^A\}$.

Outcome $j \to$ post-measurement state:

$$\tilde{\rho}_{A^n} \cong \rho_{V_j^A} \otimes \tau_{W_j^A} \quad \text{on } V_j^A \otimes W_j^A$$

Purification:

max. ent. state

$$|\tilde{\psi}_{A^n B^n}\rangle \cong |\tilde{\psi}\rangle_{V_j^A V_j^B} \otimes |\Phi^+\rangle_{W_j^A W_j^B}$$

$$\in (V_j^A \otimes V_j^B) \otimes (W_j^A \otimes W_j^B)$$

$$\cong (V_j^A \otimes W_j^A) \otimes (V_j^B \otimes W_j^B)$$

$$\subseteq (\mathbb{C}^2)^{\otimes n} \quad \otimes (\mathbb{C}^2)^{\otimes n}$$

<u>Fact</u>: Any two purifications differ only by U_{B^n}.

E.g. use Schmidt decomp.

- $|\Phi^+\rangle_{W_j^A W_j^B}$ max ent. of dimension $2^{n(h(\hat{p}) \pm \delta)}$

$\approx 2^{n(S(\rho_A) \pm \delta)}$

WHP

$\downarrow$ LOCC

$|\Phi_2^+\rangle^{\otimes n(S(\rho_A) \pm \delta)}$

Result: $\boxed{E_D(\psi) \geq S(\rho_A) = S_E(\psi)}$

- Our protocol is universal
- More refined analysis: $|\psi_{AB}\rangle^{\otimes n} \in \mathrm{Sym}^n$
  $\rightsquigarrow$ can identify $\mathbb{C} \cdot |\Phi^+\rangle \in (W_j^A \otimes W_j^B)^{S_n}$

Optimal? Yes! "Thermodynamics argument"

- Study reverse direction:

$$|\Phi_2^+\rangle^{\otimes Rn} \xrightarrow{\text{LOCC}} \approx |\psi_{AB}\rangle^{\otimes n}$$

Optimal rate: entanglement cost E_C

- Show that $\boxed{E_C(\psi) \leq S_E(\psi)}$ $(*)$
- Combine:

$$|\Phi_2^+\rangle^{\otimes E_C(\psi)\cdot n} \longrightarrow |\psi_{AB}\rangle^{\otimes n} \longrightarrow |\Phi_2^+\rangle^{\otimes E_D(\psi)\cdot n}$$

$$\Longrightarrow S_E \geq E_C \geq E_D \geq S_E \qquad \text{SEE ABOVE}$$

$$\Longrightarrow \boxed{E_C(\psi) = E_D(\psi) = S_E(\psi)}$$

3rd + 4th operational interpretation

Discussion:

- Entanglement as a resource
- Mixed state ent. (ρ_{AB})? (Or $|\psi\rangle_{ABC}$?)

  Much more complicated! In general:

  $$E_C > E_D \quad \& \quad \neq S_E$$

  Even: $E_C > 0$, $E_D = 0$ possible ... BOUND ENTANGLEMENT