## Last time:

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j V_j \otimes \mathbb{C}^{m(n,j)}$$

$$P_j \cong \bigoplus_{j'} \delta_{jj'} \cdot I$$

$$X^{\otimes n} \cong \bigoplus_j T_X^{(n,j)} \otimes I_{\mathbb{C}^{m(n,j)}}$$

$$T_X^{(n,j)} = (\det X)^{n/2} T^{(j)}_{X/\sqrt{\det X}}$$

$$\cong (\det X)^{n/2-j} \Pi_{2j} X^{\otimes 2j} \Pi_{2j}$$

For $U \in U(2)$:

$$U^{\otimes n} \cong \bigoplus_j \boxed{T_U^{(n,j)}} \otimes I$$

COMMUTES WITH $\{P_j\}$

representation of $U(2)$: "$V_{n,j}$"

For $\pi \in S_n$? Schur's lemma

$$R_\pi \cong \bigoplus_j I \otimes \boxed{R_\pi^{(n,j)}}$$

COMMUTES WITH $\{P_j\}$

representation of $S_n$: "$W_{n,j}$"

Fact: The representations $W_{n,j}$ are irreducible!

$\longrightarrow$ $\boxed{\text{Pset}}$

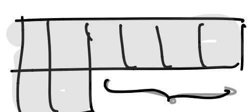
## Schur–Weyl duality:

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j \underset{\text{irrep for } U(2)}{V_{n,j}} \otimes \underset{\text{irrep for } S_n}{W_{n,j}}$$

$\hookrightarrow$ $\{P_j\}$ is finest measurement with both symmetries

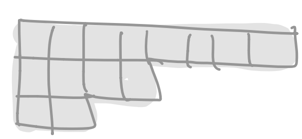
- $[R_\pi, M] = 0 \ (\forall \pi) \Longrightarrow M = \sum_i z_i X_i^{\otimes n}$
- $[U^{\otimes n}, M] = 0 \ (\forall U) \Longrightarrow M = \sum_\pi z_\pi R_\pi$

- reparametrize: YOUNG DIAGRAMS

$(n, j) \rightsquigarrow \lambda = \left(\frac{n}{2}+j, \frac{n}{2}-j\right) =$  $n$ boxes

$\hookrightarrow$ $\boxed{\hat{p} = \frac{\lambda_1}{n}, \quad 1-\hat{p} = \frac{\lambda_2}{n}}$

E.g. $W_{\square\square/\square}, W_{\square\square\square}$ are the $S_3$-irreps from L3. $\boxed{\text{Pset}}$

- generalizes to $(\mathbb{C}^d)^{\otimes n}$  $\}d$

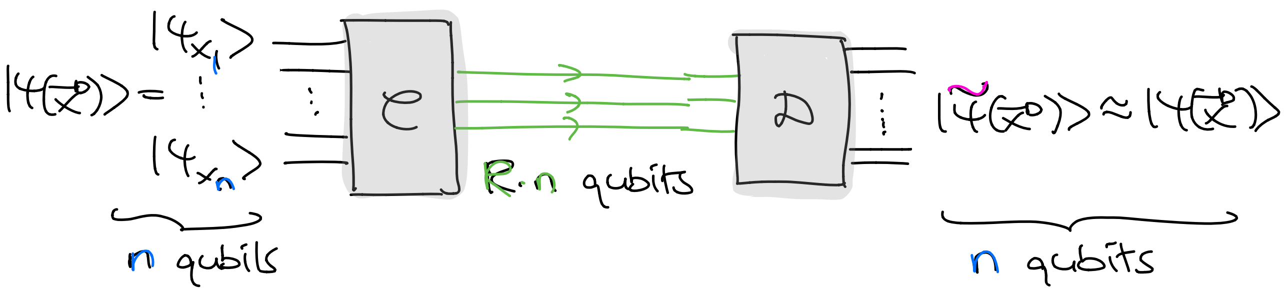
# Quantum Data Compression

**Qubit source:** $\{p_x, |\psi_x\rangle\}$

↳ emits sequences $|\psi(\vec{x})\rangle = |\psi_{x_1}\rangle \otimes \cdots \otimes |\psi_{x_n}\rangle$

$p(\vec{x}) = p_{x_1} \cdots p_{x_n}$

**Goal:** Design compressor & decompressor s.th. on average




i.e.

$$\boxed{\sum_{\vec{x}} p(\vec{x}) \cdot \mathbb{E}\, |\langle \psi(\vec{x}) | \tilde{\psi}(\vec{x}) \rangle|^2 \overset{!}{\approx} 1}$$

Since the decompressed state $|\tilde{\psi}(\vec{x})\rangle$ can be random, too!

How to solve?

Average output of source:

$$\rho = \sum_x p_x |\psi_x\rangle\langle\psi_x| \rightsquigarrow \text{eigenvalues } \{p, 1-p\}$$

- Many ensembles can give same $\rho$!
- $|\psi_x\rangle$ not necessary orthogonal $\Longrightarrow \{p_x\} \neq \{p, 1-p\}$

e.g. $\frac{1}{2}\left(|0\rangle\langle 0| + |+\rangle\langle +|\right)$ $\quad p \approx 0.85\%$ $\quad$ $\boxed{\text{Pset 1}}$

<u>Last lecture:</u> Constructed projector $\tilde{P}_n$ on $\tilde{\mathcal{H}}_n \subseteq (\mathbb{C}^2)^{\otimes n}$

s.th.

(SPECTRUM) TYPICAL PROJECTOR

$$\tilde{P}_n = \sum_{|\hat{p}-p|<\varepsilon} P_j$$

① $\mathrm{tr}\left[\rho^{\otimes n} \tilde{P}_n\right] \longrightarrow 1$

② $\dim \tilde{\mathcal{H}}_n \leq (n+1)^2 \cdot 2^{n(h(p)+\delta)}$

<u>Compression protocol:</u>

Measure observable $\tilde{P}_n$.

<u>Outcome 1</u>: Send over post-measurement state

$$|\tilde{\psi}(\vec{x})\rangle = \frac{\tilde{P}_n |\psi(\vec{x})\rangle}{\| \tilde{P}_n |\psi(\vec{x})\rangle \|} \in \tilde{\mathcal{H}}_n$$

<u>Outcome 0</u>: Send over arbitrary state

(or: fail)

(or: send over uncompressed)

$\boxed{R \approx h(p) + \delta}$ compression rate by ②

Compare coin-flip compression!

Analysis:

$$\Pr(\text{outcome } 1) = \langle \psi(\vec{x}) | \tilde{P}_n | \psi(\vec{x}) \rangle =: q(\vec{x})$$

$$\hookrightarrow \sum_{\vec{x}} p(\vec{x}) \cdot \mathbb{E} \, |\langle \psi(\vec{x}) | \tilde{\psi}(\vec{x}) \rangle|^2$$

$$\geq \sum_{\vec{x}} p(\vec{x}) \cdot q(\vec{x}) \cdot \left| \left\langle \psi(\vec{x}) \,\middle|\, \frac{\tilde{P}_n |\psi(\vec{x})\rangle}{\underbrace{\| \tilde{P}_n |\psi(\vec{x})\rangle \|}_{\sqrt{q(\vec{x})}}} \right\rangle \right|^2$$

$$= \sum_{\vec{x}} p(\vec{x}) \cdot q(\vec{x})^2 \geq \left[ \sum_{\vec{x}} p(\vec{x}) q(\vec{x}) \right]^2$$

But:

$$\sum_{\vec{x}} p(\vec{x}) q(\vec{x}) = \operatorname{tr}\Big[ \underbrace{\Big( \sum_{\vec{x}} p(\vec{x}) |\psi_{\vec{x}}\rangle\langle \psi_{\vec{x}}| \Big)}_{\rho^{\otimes n}} \tilde{P}_n \Big]$$

$$= \operatorname{tr}[\rho^{\otimes n} \tilde{P}_n] \xrightarrow{\text{by } ①} 1$$

SUCCESS!

- optimal compression rate: von Neumann entropy

$$S(\rho) := -\operatorname{tr}[\rho \cdot \log \rho] = h(p)$$

Operational interpretation

- our protocol works for all qubit sources w/ eigenvalues $\{p, 1-p\}$

Pset: universal compressor at rate $S_0$ $\forall \rho:$ $S(\rho) < S_0$

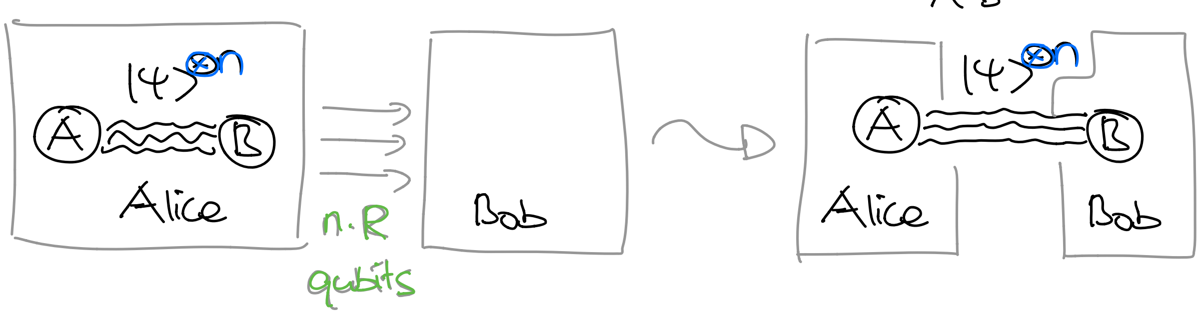
## Compression & Entanglement

density ops also appear as reduced density ops

$$\rho_B = |\psi_{AB}\rangle$$

Goal: Send our B-system to Bob at minimal qubit cost:

$|\tilde{\psi}_{A^n B^n}\rangle \approx$




"quantum state transfer"