Mixed state entanglement, monogamy of entanglement

Lecture 4 *Michael Walter, Stanford University*

These lecture notes are not proof-read and are offered for your convenience only. They include additional detail and references to supplementary reading material. I would be grateful if you email me about any mistakes and typos that you find.

Monogamy of entanglement is the idea that if two systems are strongly entangled then each of them cannot be entangled very much with other systems. For example, suppose that

$$\rho_{AB} = |\Psi\rangle\langle\Psi|_{AB}$$

where $|\Psi\rangle_{AB}$ is in a pure state – say, a maximally entangled state. Since ρ_{AB} is pure, any extension ρ_{ABC} must factorize,

$$\rho_{ABC} = \rho_{AB} \otimes \rho_C,$$

as we discussed at the end of lecture 3. Thus A and B are both completely uncorrelated with C (fig. 3). In particular, $\rho_{AC} = \rho_A \otimes \rho_C$ and $\rho_{BC} = \rho_B \otimes \rho_C$ are product states.

Remark. *While correct, the above analysis should perhaps be taken with a grain of salt. Since it only relied on ρ_{AB} being in a pure state, it is also applicable to, say, $\psi_{AB} = |0\rangle_A \otimes |0\rangle_B$ – which is a product state, not an entangled state! Nevertheless, the conclusion remains that also in this case ρ_{AC} and ρ_{BC} have to product states. However, this is a consequence of $\rho_A = |0\rangle\langle 0|_A$ and $\rho_B = |0\rangle\langle 0|_B$ being pure, not of entanglement between A and B.*

Does monogamy hold more generally and can it be made quantitative? Indeed this is possible – and we will see that symmetry is the key.

4.1 Mixed state entanglement

First, though, we will have to talk about what it means for a quantum state to be entangled. For pure states $|\psi\rangle_{AB}$, the answer is simple: A state is entangled if and only if is *not* a tensor product,

$$|\psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\psi\rangle_B\,.$$

For mixed states, however, there are non-product quantum states that should nevertheless not be considered entangled.

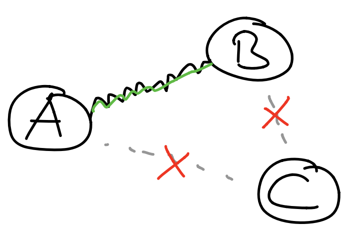

Figure 3: Illustration of monogamy of entanglement.

Example 4.1 (Classical joint distributions). *Let $p(x,y)$ be a probability distribution of two random variables. Following* (3.6)*, we construct a corresponding density operator*

$$\rho_{AB} = \sum_{x,y} p(x,y) \left|xy\right\rangle_{AB} \left\langle xy\right|_{AB} = \sum_{x,y} p(x,y) \left|x\right\rangle \left\langle x\right|_{A} \otimes \left|y\right\rangle \left\langle y\right|_{B}.$$

In general, ρ_{AB} is not a product state (indeed, this is only the case if the random variables are statistically independent). Yet this corresponds to classical correlations, not to quantum entanglement. For example, if Alice and Bob know the outcome of a fair coin flip, their state would be described by the density operator

$$\rho_{AB} = \frac{1}{2} \left(\left|00\right\rangle \left\langle 00\right|_{AB} + \left|11\right\rangle \left\langle 11\right|_{AB} \right),$$

that is not of product form.

This suggests the following general definition: We say that a quantum state ρ_{AB} is *entangled* if it is *not* a mixture of product states:

$$\rho_{AB} \neq \sum_{i} p_i \rho_A^{(i)} \otimes \rho_B^{(i)}. \tag{4.1}$$

Here, $\{p_i\}$ is an arbitrary probability distribution and the $\rho_A^{(i)}$ and $\rho_B^{(i)}$. We say that states of the right-hand side form are *separable*, or simply *unentangled.* If $\rho_{AB} = \left|\psi\right\rangle \left\langle\psi\right|_{AB}$ is a pure state then it is separable exactly if it is a product, $\left|\psi\right\rangle_{AB} = \left|\psi\right\rangle_A \otimes \left|\psi\right\rangle_B$.

Remark. *There are more separable states than the classical states in example 4.1. This is because we do not demand the operators $\{\rho_A^{(i)}\}$ and $\{\rho_B^{(i)}\}$ in eq.* (4.1) *are orthogonal.*

Separable states have a pleasant operational interpretation. They are the largest class of quantum states σ_{AB} that can be created by Alice and Bob in their laboratories if allow Alice and Bob to perform arbitrary quantum operations in their laboratory but restrict their communication with each other to be classical.

Let us denote the set of all density operators on $\mathcal{H}_A \otimes \mathcal{H}_B$ by

$$Q_{AB} = \{\rho_{AB} \geq 0, \operatorname{tr} \rho_{AB} = 1\}$$

and the subset of separable states by

$$SEP_{AB} = \{\rho_{AB} \text{separable}\}.$$

Both sets are *convex.* As a consequence of SEP_{AB} being convex, it can be fully characterized by separating hyperplanes, i.e., hyperplanes that contain all separable state on one side (fig. 4). These hyperplanes gives rise to *entanglement witness* – one-sided tests that can be used to certify that a state is entangled. You will explore them in problem 2.4.

Yet, it is unfortunately a difficult problem to decide if a mixed state is entangled or not. In fact, the problem of deciding whether a given quantum state ρ_{AB} is separable is *NP-hard.* This implies that we are unlikely to ever find an efficient (polynomial-time) algorithm. In practice, the situation is less bleak since we have ways of testing whe a quantum state is approximately separable (see below).

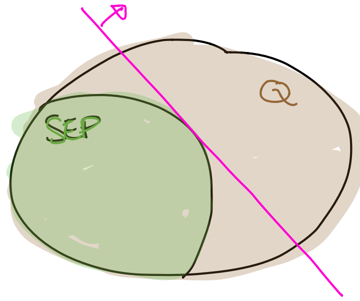

Figure 4: The set of separable states SEP is a convex subset of the set of all quantum states Q. Hyperplanes (such as the pink one) that contain all separable states on one side give rise to entanglement witnesses.

4.2 Monogamy and symmetry

We are now ready to study the monogamy of entanglement in more detail. We will consider two situations where we would expect monogamy to play a role:

De Finetti theorem

First, consider a permutation-symmetric state

$$|\Psi\rangle_{A_1\ldots A_n} \in \mathrm{Sym}^n(\mathbb{C}^d).$$

Note that all the reduced density matrices $\rho_{A_iA_j}$ are the same. Thus, every pair of particles is entangled equally, and so we would expect that by monogamy they therefore are not entangled "very much" (fig. 5, (a)).

The *quantum de Finetti theorem* (König and Renner, 2005) asserts that our expectation is indeed correct:

$$\rho_{A_1\ldots A_k} \approx \int d\psi\, p(\psi)\, |\psi\rangle^{\otimes k} \langle\psi|^{\otimes k} \tag{4.2}$$

as long as $k \ll (n-k)/d$. Here, $p(\psi)$ is some probability density over the set of pure states that depends on the state ρ. In particular, $\rho_{A_1A_2}$ is approximately a mixture of product states for large n.

Example (Warning)**.** *The GHZ state* $|\gamma\rangle_{A_1A_2A_3} = (|000\rangle + |111\rangle)/\sqrt{2}$ *is a state in the symmetric subspace* $\mathrm{Sym}^3(\mathbb{C}^2)$*. Note that, e.g., the first particle is maximally entangled with the other two – so clearly it is* not *true that permutation symmetric states are unentangled. However, if we look at the reduced state of two particles then we find*

$$\rho_{A_1A_2} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) = \frac{1}{2}|0\rangle^{\otimes 2}\langle 0|^{\otimes 2} + \frac{1}{2}|1\rangle^{\otimes 2}\langle 1|^{\otimes 2}.$$

Note that $\rho_{A_1A_2}$ *is a* mixture *of product states. This shows that the partial trace is indeed necessary.*

Permutation symmetric states arise naturally in *mean-field systems*. The ground state $|E_0\rangle$ of a mean-field Hamiltonian $H = \sum_{1\leq i<j\leq n} h_{ij}$ is necessarily in the symmetric subspace – provided that the ground space is nondegenerate and that n is larger than the single-particle Hilbert space. Thus, the de Finetti theorem shows that, locally, ground states of mean field systems look like mixtures of product states – a property that is highly useful for their analysis. For example, it allows us to use the density $p(\psi)$ as a variational ansatz.

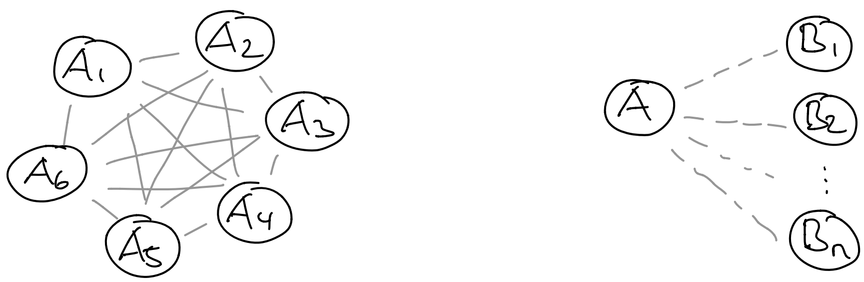

Figure 5: (a) In a permutation symmetric state, any pair of particles is entangled in the same way and should therefore not be entangled very much. (b) Similarly, if Alice is entangled with many Bobs in the same way then she is not entangled very much with each of them.

Extendibility hierarchy

A closely related situation is the following: Suppose that ρ_{AB} is a quantum state that has an extension $\rho_{AB_1 \dots B_n}$ such that

$$\rho_{AB_i} = \rho_{AB} \qquad (\forall i, j)$$

(fig. 5, (b)). We say that ρ_{AB} has an *n-extension*. Thus A is equally entangled with all B_i and so we would expect that ρ_{AB} is not entangled "very much". Indeed, it is true that, for large n,

$$\rho_{AB} \approx \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)},$$

i.e., ρ_{AB} is again approximately a mixture of product states.

In contrast to situation (1), however, there is no longer a symmetry requirement between A and B, i.e., this reasoning applies to general states ρ_{AB}. It turns out that one in this way obtains a hierarchy of efficient approximates test for separability (Doherty et al., 2002, 2004). Indeed, as you will discuss in problem 2.5, if a state ρ_{AB} is n-extendible then it is $O(1/n)$-close to being a separable state (fig. 6).

4.3 The trace distance between quantum states

Before we proceed, we should make more precise what we meant when we wrote "≈" above. Let ρ and σ be two density operators on some Hilbert space $\mathcal{H}$. We define their *trace distance* to be

$$T(\rho, \sigma) := \max_{0 \leq Q \leq \mathbb{1}_{\mathcal{H}}} \operatorname{tr}[Q(\rho - \sigma)].$$

The trace distance is a metric, and so in particular satisfies the triangle inequality. It has the following alternative expression

$$T(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_1,$$

where we used the *1-norm*, which for general Hermitian operators Δ with spectral decomposition $\Delta = \sum_i \lambda_i |e_i\rangle \langle e_i|$ is defined by $\|\Delta\|_1 = \sum_i |\lambda_i|$. The trace distance has a natural operational interpretation in terms of the optimal probability of distinguishing ρ and σ by a POVM measurement. You discussed the trace distance in problem 1.3 in the special case of pure states, but the above conclusions hold

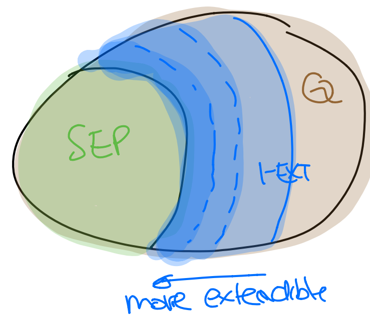

Figure 6: The extendibility hierarchy: If a state is n extendible then it is $O(1/n)$-close to being separable.

in general. There, you also proved that, for pure states $\rho = |\phi\rangle\langle\phi|$ and $\sigma = |\psi\rangle\langle\psi|$, the trace distance and overlap are related by the following formula:

$$T(\rho,\sigma) = \sqrt{1 - |\langle\phi|\psi\rangle|^2} \tag{4.3}$$

Remark. *If X is an arbitrary observable then*

$$|\mathrm{tr}[H\rho] - \mathrm{tr}[H\sigma]| \leq 2T(\rho,\sigma)\|H\|_\infty, \tag{4.4}$$

where $\|H\|_\infty$ denotes the operator norm *of H, defined as the maximal absolute value of all eigenvalues of H. Indeed, we can always write $H = Q - Q' <$ where $0 \leq Q, Q' \leq \|H\|_\infty$, and so*

$$|\mathrm{tr}[H\rho] - \mathrm{tr}[H\sigma]| \leq |\mathrm{tr}[Q\rho] - \mathrm{tr}[Q\sigma]| + |\mathrm{tr}[Q'\rho] - \mathrm{tr}[Q'\sigma]| \leq 2\|H\|_\infty T(\rho,\sigma).$$

Equation (4.4) quantifies the difference in expectation values for states with small trace distance. (Note that this gap gap can be arbitrarily large since we can always rescale our observable. This is reflected by the factor $\|H\|_\infty$.)rescale our observables.

4.4 The quantum de Finetti theorem

We will now prove the de Finetti theorem (4.2), following Brandao et al. (2016). Let

$$|\Phi\rangle_{A_1\ldots A_n} \in \mathrm{Sym}^n(\mathbb{C}^d),$$

where n is the number of particles and d the dimension of the single-particle Hilbert space.

The basic idea is the following: Suppose that we measure with the uniform POVM (2.8) on the last $n-k$ systems of $\rho = |\Phi\rangle\langle\Phi|$. Then, if the measurement outcome is some $|\psi\rangle$, we would expect that the first k systems are likewise in the state $|\psi\rangle^{\otimes k}$, at least on average, since the overall state is permutation symmetric among all n subsystems.

Let us try to implement this idea. Since $|\Phi\rangle \in \mathrm{Sym}^n(\mathbb{C}^d)$, it is in particular symmetric under permutations of the last $n-k$ subsystems. Hence, $|\Phi\rangle = (\mathbb{1}_k \otimes \Pi_{n-k})|\Phi\rangle$, and so

$$\begin{aligned}
\rho_{A_1\ldots A_k} &= \mathrm{tr}_{A_{k+1}\ldots A_n}[|\Phi\rangle\langle\Phi|] = \mathrm{tr}_{A_{k+1}\ldots A_n}[(\mathbb{1}_k \otimes \Pi_{n-k})|\Phi\rangle\langle\Phi|] \\
&= \binom{n-k+d-1}{n-k} \int d\psi\, (\mathbb{1}_k \otimes \langle\psi|^{\otimes(n-k)})|\Phi\rangle\langle\Phi|(\mathbb{1}_k \otimes |\psi\rangle^{\otimes(n-k)}) \\
&= \int d\psi p(\psi)\, |V_\psi\rangle\langle V_\psi|.
\end{aligned}$$

In the second to last step, we have inserted the resolution of identity (2.6), and in the last step, we have introduced introduced unit vectors $|V_\psi\rangle$ and numbers $p(\psi) \geq 0$ such that

$$\sqrt{p(\psi)}\,|V_\psi\rangle = \binom{n-k+d-1}{n-k}^{1/2} (\mathbb{1}_k \otimes \langle\psi|^{\otimes(n-k)})\,|\Phi\rangle\,. \tag{4.5}$$

Note that $p(\psi)$ is a probability density. Indeed, $\int d\psi\, p(\psi) = \operatorname{tr}\rho = 1$, since the overall state is normalized. We would now like to prove that

$$\rho_{A_1\ldots A_k} = \int d\psi\, p(\psi)\; |V_\psi\rangle\langle V_\psi| \approx \int d\psi\, p(\psi)\; |\psi\rangle^{\otimes k}\langle\psi|^{\otimes k} =: \widetilde{\rho}_{A_1\ldots A_k}, \tag{4.6}$$

based on the intuition expressed above that on average the post-measurement states $|V_\psi\rangle$ are close to $|\psi\rangle^{\otimes k}$. Let us first consider the average overlap:

$$\begin{aligned}
&\int d\psi\, p(\psi)\, |\langle V_\psi|\psi^{\otimes k}\rangle|^2 = \int d\psi\, p(\psi)\; \langle V_\psi|\psi^{\otimes k}\rangle\langle\psi^{\otimes k}|V_\psi\rangle \\
&= \binom{n-k+d-1}{n-k}\int d\psi\; \langle\Phi|\psi^{\otimes n}\rangle\langle\psi^{\otimes n}|\,|\Phi\rangle\rangle = \binom{n-k+d-1}{n-k}\binom{n+d-1}{n}^{-1}\underbrace{\langle\Phi|\Pi_n|\Phi\rangle}_{=1} \\
&= \binom{n-k+d-1}{n-k}\binom{n+d-1}{n}^{-1} \geq 1 - \frac{kd}{n-k}.
\end{aligned}$$

In the second step, we inserted the definition of $|V_\psi\rangle$ from eq. (4.5). And the last inequality is precisely (2.9), since there we bounded precisely the ratio of binomial coefficients that we are interested in here (with $n \mapsto n+k$).

It remains to show that the two states ρ and $\widetilde{\rho}$ in eq. (4.6) are close in trace distance. Indeed,

$$\begin{aligned}
&T(\rho_{A_1\ldots A_k}, \widetilde{\rho}_{A_1\ldots A_k}) \leq \int d\psi\, p(\psi) T(|V_\psi\rangle\langle V_\psi|, |\psi\rangle^{\otimes k}\langle\psi|^{\otimes k}) = \int d\psi\sqrt{1 - |\langle V_\psi|\psi^{\otimes k}\rangle|^2} \\
&\leq \sqrt{\int d\psi\,\left(1 - |\langle V_\psi|\psi^{\otimes k}\rangle|^2\right)} = \sqrt{1 - \int d\psi\, |\langle V_\psi|\psi^{\otimes k}\rangle|^2} \leq \sqrt{\frac{kd}{n-k}}.
\end{aligned}$$

Here, we first applied the triangle inequality, then we used the relationship between trace distance and fidelity for pure states in eq. (1.2), and the next inequality i Jensen's inequality for the square root function, which is concave. Thus we have proved the de Finetti theorem (4.2):

$$\rho_{A_1\ldots A_k} \approx \int d(\psi)\; |\psi\rangle^{\otimes k}\langle\psi|^{\otimes k}$$

up to error $\sqrt{kd/(n-k)}$ in trace distance. Explicitly, the density $p(\psi)$ that we used in our proof is given by $\langle\Phi|\mathbb{1}_k \otimes Q_\psi|\Phi\rangle$, where $\{Q_\psi\}$ is the uniform POVM (2.8).

Beyond the symmetric subspace

Our intuition behind the de Finetti theorem only relied on the fact that the reduced density matrices were all the same. But this is a feature that states on the symmetric subspace share with arbitrary *permutation-invariant* states, i.e., states that satisfy

$$[R_\pi, \rho_{A_1\ldots A_n}] = 0, \quad \text{or} \quad R_\pi\rho_{A_1\ldots A_n} = \rho_{A_1\ldots A_n}R_\pi$$

for all $\pi \in S_n$. Examples of permutation-invariant states are states on the *anti*symmetric subspace, or tensor powers of mixed states such as $\rho^{\otimes n}$, which we will study in more detail next week.

A useful fact is that any permutation-invariant state $\rho_{A_1 \dots A_n}$ has a purification on a symmetric subspace: That is, there exists a pure state $|\Phi\rangle_{(A_1 B_1)\dots(A_n B_n)} \in \mathrm{Sym}^n(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\mathcal{H}_B$ is some auxiliary space, such that $\rho_{(A_1 B_1)\dots(A_n B_n)} = |\Phi\rangle\langle\Phi|$ is an extension of $\rho_{A_1\dots A_n}$. The auxiliary space $\mathcal{H}_B$ can be chosen of the same dimension as $\mathcal{H}_A$.

If we apply the de Finetti theorem to such a purification, we find that

$$\rho_{(A_1 B_1)\dots(A_k B_k)} \approx \int d\psi_{AB}\, p(\psi_{AB})\, |\psi\rangle_{AB}^{\otimes k} \langle\psi|_{AB}^{\otimes k}$$

up to error $d^2 k/(n-k)$, since now the single-particle Hilbert space has dimension $\dim \mathcal{H}_A \otimes \mathcal{H}_B = d^2$. If we take a partial trace over the B systems, we obtain a mixture of product states (which can now be mixed):

$$\rho_{A_1\dots A_k} \approx \int d\psi_{AB}\, p(\psi_{AB})\, \mathrm{tr}_B[|\psi\rangle\langle\psi|_{AB}]^{\otimes k}$$

Moreover, the trace distance never increases when we take the partial trace. Thus we have proved the following: If $\rho_{A_1\dots A_n}$ is a permutation-invariant state on $(\mathbb{C}^d)^{\otimes k}$ then its reduced density matrices can be approximated by mixtures of product states

$$\rho_{A_1\dots A_k} \approx \int d\mu(\rho)\, \rho^{\otimes k}$$

up to error $d^2/(n-k)$ in trace distance. Here, $d\mu$ is some probability measure on the space of mixed states that depends on the state ρ.

Nowadays, there are many variants of the de Finetti theorem that quantify the monogamy of entanglement in interesting and useful ways. Surveying some of them could make for an interesting course project.

Bibliography

Robert König and Renato Renner. A de finetti representation for finite symmetric quantum states, *Journal of Mathematical physics*, 46(12):122108, 2005. arXiv:quant-ph/0410229.

Andrew C Doherty, Pablo A Parrilo, and Federico M Spedalieri. Distinguishing separable and entangled states, *Physical Review Letters*, 88(18):187904, 2002. arXiv:quant-ph/0112007.

Andrew C Doherty, Pablo A Parrilo, and Federico M Spedalieri. Complete family of separability criteria, *Physical Review A*, 69(2):022308, 2004. arXiv:quant-ph/0308032.

Fernando GSL Brandao, Matthias Christandl, Aram W Harrow, and Michael Walter. The Mathematics of Entanglement. 2016. arXiv:1604.01790.