Last week's claim: Projector Π_n on $\mathrm{Sym}^n(\mathbb{C}^d)$ given by

$$\Pi_n' = \binom{n+d-1}{n} \int d\psi \, |\psi\rangle^{\otimes n}\langle\psi|^{\otimes n}$$

measure invariant under $|\psi\rangle \mapsto U|\psi\rangle$

$$\Rightarrow \boxed{U^{\otimes n}\,\Pi_n'\,U^{\dagger\otimes n} = \Pi_n'} \qquad \circledast$$

Representation Theory Primer

Group G: $g,h \in G \Rightarrow gh^{-1} \in G$, $gg^{-1} = g^{-1}g = 1$

(Unitary) representation:

- Hilbert Space $\mathcal{H}$
- unitary operators $\{R_g\}_{g\in G}$ on $\mathcal{H}$

$$R_1 = I_{\mathcal{H}} \quad \& \quad R_{gh} = R_g R_h$$

To understand a representation, want to decompose it into its smallest building blocks...

Invariant subspace: $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ s.th.

$$|\psi\rangle \in \tilde{\mathcal{H}} \Rightarrow R_g|\psi\rangle \in \tilde{\mathcal{H}} \quad (\forall g)$$

$\Rightarrow \tilde{\mathcal{H}}^{\perp}$ is also invariant subspace:

$$\mathcal{H} = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}^{\perp}$$

$$R_g = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

"smalle" representations

$\rightsquigarrow$ "finest" <u>decomposition</u>:

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$$

$$\uparrow \qquad \cdots \qquad \uparrow$$

<u>irreducible representation</u> ("irreps")

RT tells us how to decompose & what the irreps are!

<u>Ex</u>:
- $G = SU(2)$: spin-j irreps V_j w/ basis $|j,m\rangle$
- $G = S_3$: Three irreps:

$$\mathcal{H}_{\square\square\square} = \mathbb{C}|0\rangle \qquad R_\pi |0\rangle = |0\rangle \qquad \text{"trivial"}$$

$$\mathcal{H}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \mathbb{C}|0\rangle \qquad R_\pi |0\rangle = \operatorname{sign}(\pi)|0\rangle$$

$$\mathcal{H}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = \{\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle : \alpha+\beta+\gamma = 0\} \subseteq \mathbb{C}^3$$

$$R_\pi |j\rangle = |\pi(j)\rangle \qquad \longrightarrow \boxed{\text{PSET}}$$

Note: $\mathcal{H}^{\perp}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = \mathbb{C}\cdot(|0\rangle + |1\rangle + |2\rangle) \cong \mathcal{H}_{\square\square\square}$

$$\Longrightarrow \mathbb{C}^3 \cong \mathcal{H}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} \oplus \mathcal{H}_{\square\square\square}$$

$\uparrow$ What does this mean?

How to compare representations?

Intertwiner: $J : \mathcal{H} \to \mathcal{H}'$ s.th. $J R_g = R'_g J$

if invertible: $\boxed{J R_g J^{-1} = R'_g}$ base change

⇝ $\mathcal{H}, \mathcal{H}'$ are equivalent, "$\mathcal{H} \cong \mathcal{H}'$"

e.g., can write

$$\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_m \cong V_{j_1} \oplus \dots \oplus V_{j_m}$$

$$\cong \bigoplus_j V_j \otimes \mathbb{C}^{n_j} \quad \text{where } V_j \not\cong V_{j'}$$

Schur's Lemma: If $\mathcal{H}, \mathcal{H}'$ irreps, $J : \mathcal{H} \to \tilde{\mathcal{H}}$ interw.

① Either J invertible or $J = 0$!

② If $\mathcal{H} = \mathcal{H}'$ & $R_g = R'_g$: $\boxed{J \propto I_{\mathcal{H}}}$

Let's prove our claim:

$(\mathbb{C}^d)^{\otimes n}$ is representation of S_n & of $U(d)$:

- $R_\pi(|\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle) = |\psi_{\pi^{-1}(1)}\rangle \otimes \dots \otimes |\psi_{\pi^{-1}(n)}\rangle$
- $T_U = U^{\otimes n} = U \otimes \dots \otimes U$

$\boxed{[R_\pi, T_U] = 0}$!

$\rightsquigarrow$ $\mathrm{Sym}^n(\mathbb{C}^d)$ is invariant subspace of $U(d)$

$R_\pi|\Phi\rangle = |\Phi\rangle \Rightarrow R_\pi(T_u|\Phi\rangle) = T_u(R_\pi|\Phi\rangle) = T_u|\Phi\rangle$

In fact: $\mathrm{Sym}^n(\mathbb{C}^d)$ is irrep of $U(d)$! $\rightarrow$ $\boxed{\text{PSET}}$

OTOH: ⊛ says that Π'_n is intertwiner on $\mathrm{Sym}^n(\mathbb{C}^d)$

$$\xRightarrow{\text{SCHUR}} \Pi'_n \propto \Pi_n \xRightarrow[\text{trace}]{\text{same}} \Pi'_n = \Pi_n . \qquad \square$$

Density Operator Primer

So far: $|\psi\rangle \in \mathcal{H}$ "pure state"

Ensemble $\{p_i, |\psi_i\rangle\}$, X observable:

$$\langle X\rangle = \sum_i p_i \langle\psi_i|X|\psi_i\rangle$$

$$= \sum_i p_i \operatorname{tr}\left[|\psi_i\rangle\langle\psi_i| X\right]$$

$$= \operatorname{tr}\Big[\underbrace{\sum_i p_i |\psi_i\rangle\langle\psi_i|}_{=:\,\rho} \cdot X\Big]$$

$$\boxed{\begin{array}{l} \bullet\ \rho \geq 0 \\ \bullet\ \operatorname{tr}\rho = 1 \end{array}}$$

density operator, "mixed state"

Born's rule: $\Pr(\text{outcome } x) = \operatorname{tr} \rho P_x$

Pure state: $\rho = |\psi\rangle\langle\psi| \iff \operatorname{rk}(\rho) = 1 \iff \rho^2 = \rho.$

WARNING: Ensemble is **not** unique!

$$\frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}\left(|+\rangle\langle +| + |-\rangle\langle -|\right)$$

→ Statistical ensembles (Gibbs states etc.), classical probability: $\{p_x, |x\rangle\} \to \sum_x p_x |x\rangle\langle x|$ noisy sources, ... but also subsystems!

ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, X_A observable:

$$\langle X_A\rangle = \mathrm{tr}\left[\rho_{AB}(X_A\otimes I_B)\right]$$

$$= \sum_{a,b} \underbrace{\langle ab|\rho_{AB}(X_A\otimes I_B)|ab\rangle}_{\langle a|(I_A\otimes\langle b|)\rho_{AB}(I_A\otimes|b\rangle)X_A|a\rangle}$$

$$= \mathrm{tr}\Big[\underbrace{\sum_b (I_A\otimes\langle b|)\rho_{AB}(I_A\otimes|b\rangle)}_{=:\ \mathrm{tr}_B[\rho_{AB}]\ =\ \rho_A}\ X_A\Big]$$

partial trace   reduced density operator

makes sense for arbitrary operators M_{AB}

$$\Rightarrow \mathrm{tr}\left[M_{AB}(X_A\otimes I_B)\right] = \mathrm{tr}\left[\mathrm{tr}_B[M_{AB}]\cdot X_A\right]$$

Rule:
- $\mathrm{tr}_B[M_A\otimes N_B] = M_A\cdot\mathrm{tr}[N_B]$ hence "partial" trace

WARNING: Even if ρ_{AB} pure, ρ_A can be mixed!

$$|\psi_{AB}\rangle = \sqrt{\frac{1}{2}}\left(|00\rangle + |11\rangle\right)$$

$$\Rightarrow \rho_{AB} = \frac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right)$$

$$= \frac{1}{2}\Big(\underbrace{|00\rangle\langle 00|}_{=|0\rangle\langle 0|\otimes|0\rangle\langle 0|} + \underbrace{|00\rangle\langle 11|}_{=|0\rangle\langle 1|\otimes|0\rangle\langle 1|} + |11\rangle\langle 00| + |11\rangle\langle 11|\Big)$$

$$\Rightarrow \rho_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Signature of entanglement for pure states!

Any $|\psi_{AB}\rangle$ has a Schmidt decomposition:

$\boxed{\text{Pset}}$

$$|\psi_{AB}\rangle = \sum_i \underset{\overset{\uparrow}{>0}}{s_i}\ \underset{\overset{\uparrow}{\text{ortho-normal}}}{|e_i\rangle_A} \otimes \underset{\overset{\uparrow}{\text{ortho-normal}}}{|f_i\rangle_B}$$

$\boxed{\text{THE FULL STORY}}$

$$\rho_A = \sum_i |s_i|^2 |e_i\rangle\langle e_i|_A \qquad \rho_B = \sum_i |s_i|^2 |f_i\rangle\langle f_i|_B$$

- Any ρ_A has purification $|\psi_{AB}\rangle$:

$$\rho_A = \mathrm{tr}_B\left[|\psi_{AB}\rangle\langle\psi_{AB}|\right]$$

"church of larger H. space"

$\boxed{\text{PSET}}$

- ρ_A pure $\Rightarrow$ $\rho_{AB} = \rho_A \otimes \rho_B$

whether ρ_{AB} pure or mixed