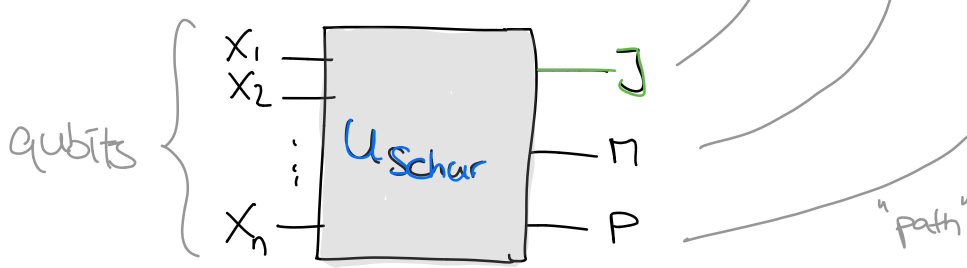
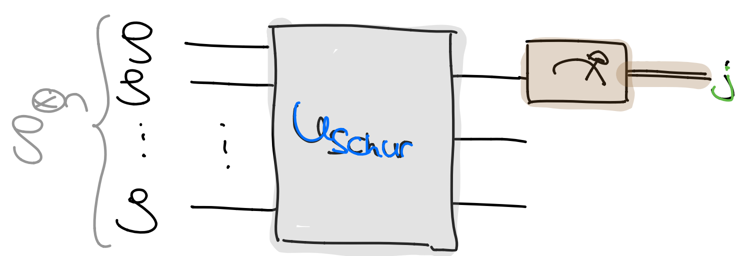
Goal: A quantum circuit for

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j V_j \otimes \mathbb{C}^{m(n,j)} \subseteq \mathbb{C}^n \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{2^n}$$



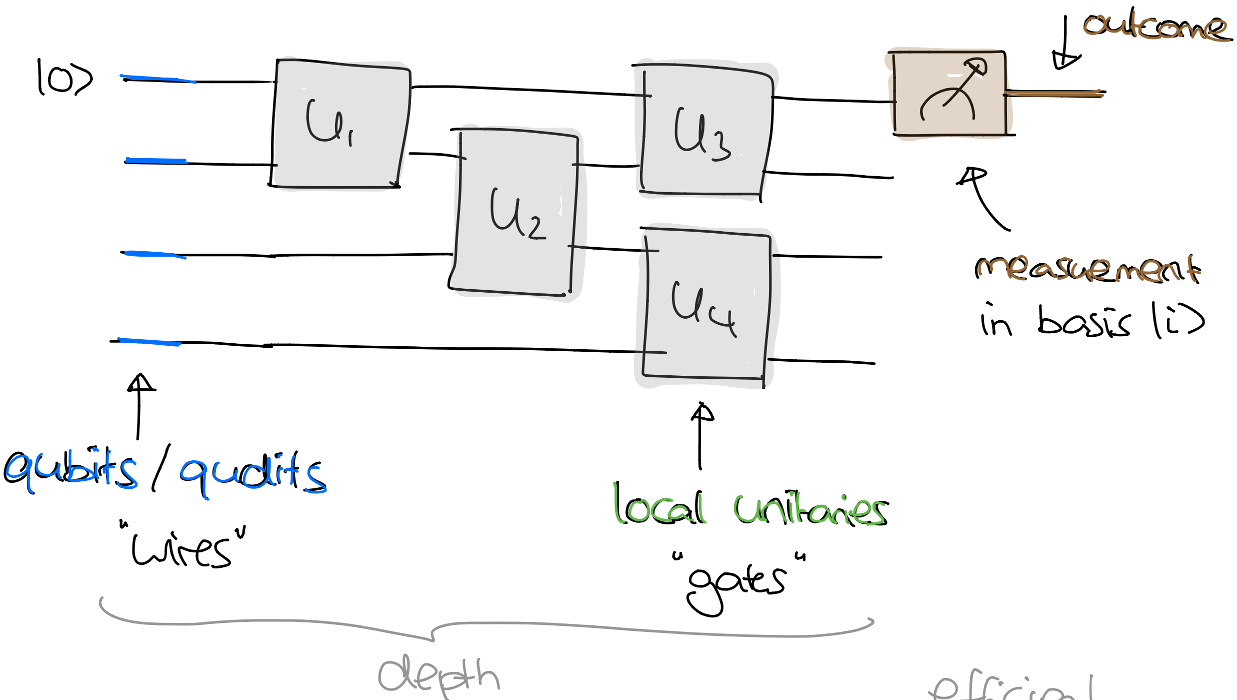
"Quantum Schur transform"

E.g., spectrum estimation measurement $\{P_j\}$:



Quantum circuits

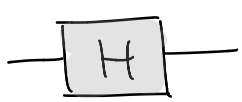
What is a quantum circuit?



efficient = small # of gates (e.g., poly(n))

gates = complexity

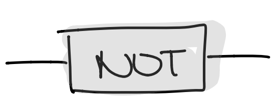
- Hadamard gate:



$H|0\rangle = |+\rangle$
$\quad|1\rangle = |-\rangle$

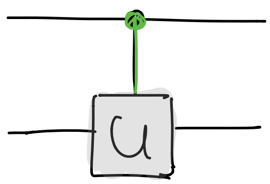
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- NOT/X gate:



$X|0\rangle = |1\rangle$
$\quad|1\rangle = |0\rangle$

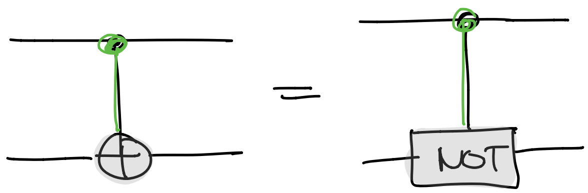
- Controlled unitary:



$CU(|0\rangle \otimes |\psi\rangle) = |0\rangle \otimes |\psi\rangle$

$(|1\rangle \otimes |\psi\rangle) = |1\rangle \otimes U|\psi\rangle$

e.g. CNOT:



$CNOT\,|00\rangle = |00\rangle$

$|01\rangle = |01\rangle$

$|10\rangle = |11\rangle$

$|11\rangle = |10\rangle$

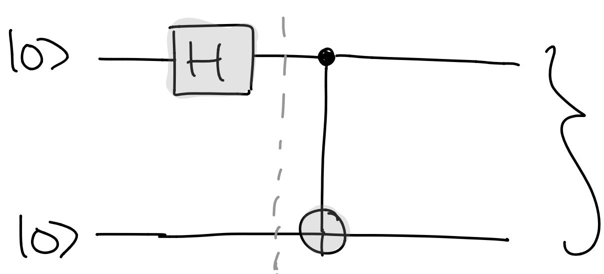
i.e.

$CNOT\,|x,y\rangle = |x, x \oplus y\rangle$

This is an "entangling gate"!

Fact: CNOT & single-qubit U's are <u>universal</u>.

Interesting circuits:



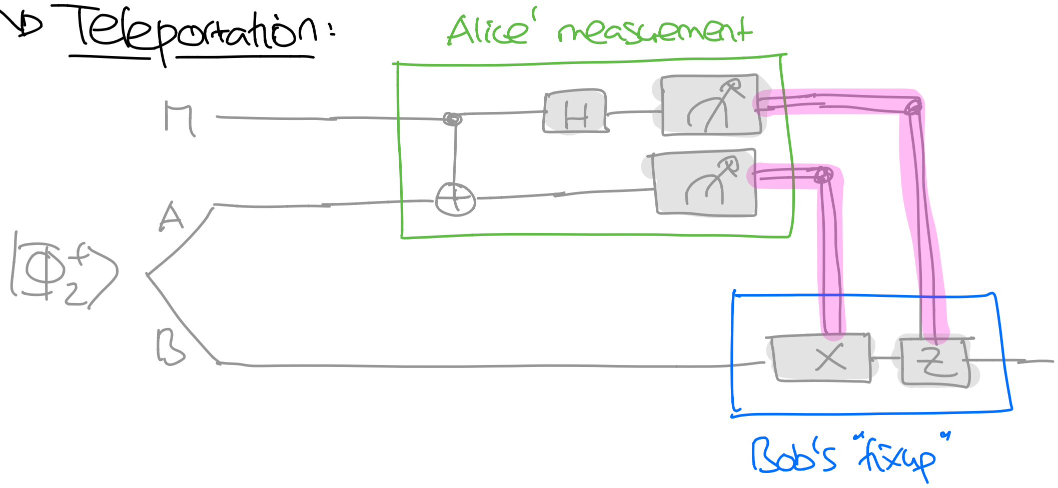
$$|\Phi^{+}_{2}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$\sqrt{\frac{1}{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$

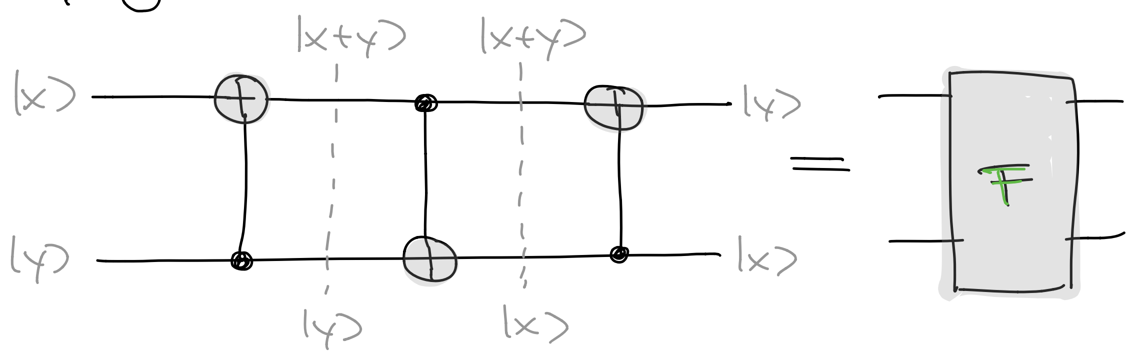
$= \sqrt{\frac{1}{2}}(|00\rangle + |10\rangle)$

NB: $|x\rangle|y\rangle \longmapsto |\phi_{k}\rangle$

→ <u>Teleportation:</u>



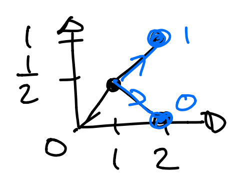
• Swap gate:

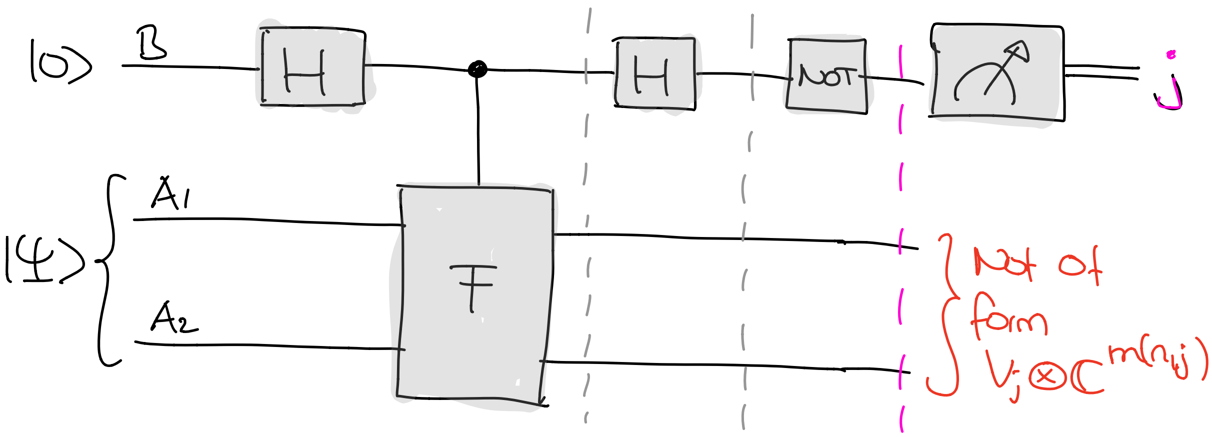


This already allows us to solve the case of 2 qubits!

<u>Warmup:</u> $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$= \underbrace{\mathrm{Sym}^2(\mathbb{C}^2)}_{\text{Spin } 1} \oplus \underbrace{\Lambda^2(\mathbb{C}^2)}_{\text{Spin } 0}$$





$$\frac{1}{\sqrt{2}}\left[|0\rangle_B \otimes |\psi\rangle_{A_1A_2} + |1\rangle_B \otimes F|\psi\rangle_{A_1A_2}\right]$$

$$\frac{1}{\sqrt{2}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\otimes|\psi\rangle + \frac{|0\rangle-|1\rangle}{\sqrt{2}}\otimes F|\psi\rangle\right)$$

$$=\frac{1}{2}\left[|0\rangle\otimes(|\psi\rangle+F|\psi\rangle) + |1\rangle\otimes(|\psi\rangle-F|\psi\rangle)\right]$$

$$= |0\rangle\otimes \underbrace{\boxed{\Pi_2|\psi\rangle}}_{\in \mathrm{Sym}^2=V_1} + |1\rangle\otimes\underbrace{\boxed{(I-\Pi_2)|\psi\rangle}}_{\in\Lambda^2=V_0}$$

Circuit implements: $\boxed{|\psi\rangle \longmapsto \sum_{j=0}^{1} |j\rangle\otimes P_j|\psi\rangle}$

i.e. $\tilde{\rho} \longmapsto \sum_{j,\tilde{j}} |j\rangle\langle\tilde{j}| \otimes P_j\,\tilde{\rho}\,P_{\tilde{j}}$

↳ $\Pr(\text{outcome } j) = \mathrm{tr}\left[P_j\,\tilde{\rho}_{A_1A_2}\right]$

Applications of this SWAP TEST:

- $\tilde{\rho} = \rho \otimes \rho$:

$$\Pr(\text{outcome } j) = \operatorname{tr}[P_j \rho^{\otimes 2}]$$

In particular:

$$\Pr(\text{outcome } 1) = \frac{1}{2}(1 + \operatorname{tr} \rho^2) \overset{!}{=} 1 \iff \rho \text{ pure}$$

↳ can estimate purity given two copies of unknown

- $\tilde{\rho} = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|$:

$$\Pr(\text{outcome } 1) = \frac{1}{2}(1 + \operatorname{tr}[|\psi\rangle\langle\psi| \cdot |\phi\rangle\langle\phi|])$$

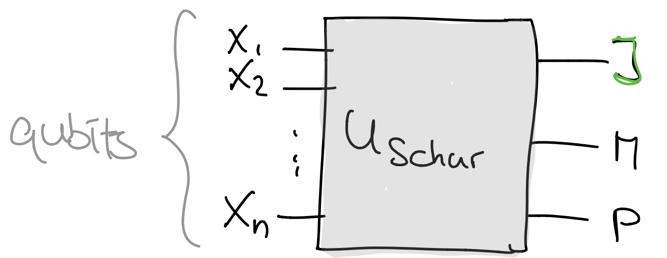
$$= \frac{1}{2}(1 + |\langle\psi|\phi\rangle|^2) \overset{!}{=} 1 \iff |\psi\rangle = e^{i\theta} |\phi\rangle$$

↳ can test equality of unknown pure states

Fun application:

A quantum circuit for Schur-Weyl duality

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_{j} V_j \otimes \mathbb{C}^{m(n,j)} \subseteq \mathbb{C}^n \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{2^n}$$



key ingredient:

$$V_j \otimes V_{\frac{1}{2}} \cong \bigoplus_{j'=j-\frac{1}{2}}^{j+\frac{1}{2}} V_{j'}$$

CG coefficients: $\langle j', m' | \left(|j,m\rangle \otimes |\tfrac{1}{2}, s\rangle \right)$

Recall: V_j has basis $|j,m\rangle$

$$\tilde{Z}\,|j,m\rangle = 2m \cdot |j,m\rangle$$

$\hookrightarrow$ $|j,m\rangle \otimes |\tfrac{1}{2}, s\rangle = \# \cdot |j+\tfrac{1}{2}, m+s\rangle + \# \cdot |j-\tfrac{1}{2}, m+s\rangle$

2x2 unitary $(s = \pm\tfrac{1}{2})$

$$(\tilde{Z} \otimes I + I \otimes Z)|j,m\rangle \otimes |\tfrac{1}{2},s\rangle$$
$$= 2(m+s)(|j,m\rangle \otimes |\tfrac{1}{2},s\rangle)$$
$$\Longrightarrow \boxed{m' = m+s}$$

How to determine #?

$$\tilde{S}_+|j,m\rangle = 2\sqrt{j(j+1) - m(m+1)}\ |j,m+1\rangle$$
$$\tilde{S}_-|j,m\rangle = 2\sqrt{j(j+1) - m(m-1)}\ |j,m-1\rangle$$

where $S_\pm = X \pm iY$.

Thus:

- $|j+\tfrac{1}{2}, j+\tfrac{1}{2}\rangle = |j,j\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$ $\quad S_-$

$$\rightsquigarrow |j+\tfrac{1}{2}, m'\rangle = \# \cdot |j,m'-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle + \# \cdot |j,m'+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

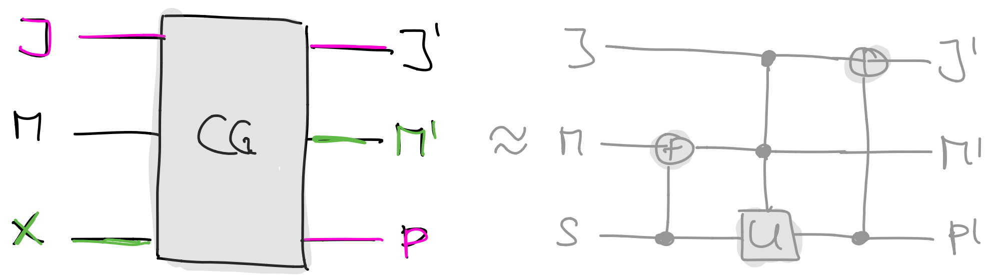
- $|j-\tfrac{1}{2}, j-\tfrac{1}{2}\rangle$ by orthogonality to $|j+\tfrac{1}{2}, j-\tfrac{1}{2}\rangle$

$$\rightsquigarrow |j-\tfrac{1}{2}, m'\rangle = \# \cdot |j,m'-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle + \# \cdot |j,m'+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

↳ invert to obtain #.

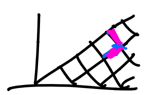
Clebsch-Gordan trafo:

ISOMETRY ($m \le j \le \frac{n}{2}$ etc.)



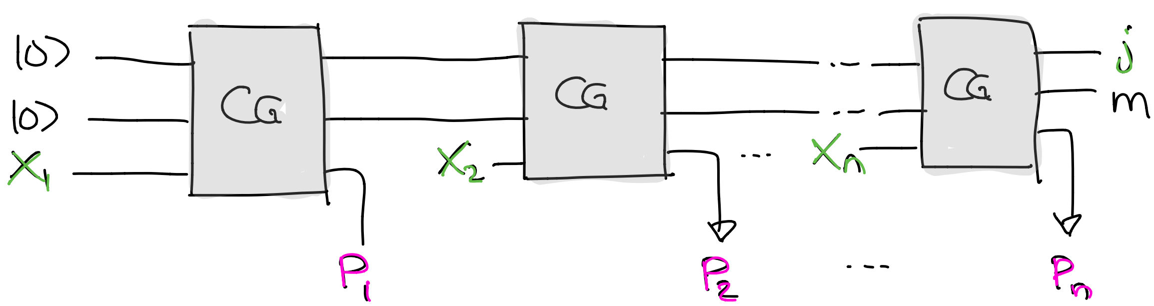
$$|j,m,x\rangle \longmapsto \#\cdot|j+\tfrac{1}{2}, m+s\rangle\otimes|+\rangle + \#\cdot|j-\tfrac{1}{2}, m+s\rangle\otimes|-\rangle \qquad \boxed{s=\tfrac{1}{2}-x}$$

Note: j part of input $\rightarrow$ need to remember where we come from



$(j+\frac{p}{2}=j')$

Schur-Weyl transform:

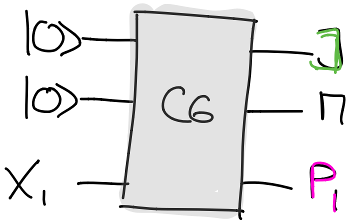


Implements

$$|x_1\rangle\cdots|x_n\rangle \in (\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j \overset{|m\rangle}{\overset{\cap}{V_j}} \otimes \overset{|\vec{p}\rangle}{\overset{\cap}{\mathbb{C}^{m(n,j)}}}$$

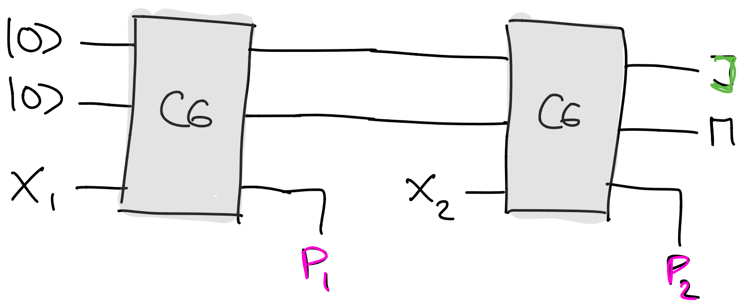
$$\subseteq \mathbb{C}^n \otimes \mathbb{C}^{n+1} \otimes \overset{\text{path}}{\boxed{(\mathbb{C}^2)^{\otimes n}}}$$

Example:



$$|0\rangle \mapsto |\tfrac{1}{2}\rangle_J \otimes |\tfrac{1}{2}\rangle_M \otimes |+\rangle_P$$

$$|1\rangle \mapsto |\tfrac{1}{2}\rangle_J \otimes |-\tfrac{1}{2}\rangle_M \otimes |+\rangle_P$$



- $|0\rangle_{X_1} |0\rangle_{X_2} \longmapsto |1\rangle_J \otimes |1\rangle_M \otimes |++\rangle_P$

- $|0\rangle_{X_1} |1\rangle_{X_2} \longmapsto \sqrt{\tfrac{1}{2}}\, |1\rangle_J \otimes |0\rangle_M \otimes |++\rangle_P$
  $|1\rangle \quad |0\rangle$
  $\qquad \pm \sqrt{\tfrac{1}{2}}\, |0\rangle_J \otimes |0\rangle_M \otimes |+-\rangle_P$

- $|1\rangle_{X_1} |1\rangle_{X_2} \longmapsto |1\rangle_J \otimes |-1\rangle_M \otimes |++\rangle_P$

$$\left(\left.\begin{matrix} |0\rangle|1\rangle \\ |1\rangle|0\rangle \end{matrix}\right\} = \sqrt{\tfrac{1}{2}}\ \frac{|01\rangle + |10\rangle}{\sqrt{2}} \pm \sqrt{\tfrac{1}{2}}\ \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right)$$