Reed-Solomon Codes

e.g. PDF417 bar code
$q=929, \ \alpha=3, \ T=4$

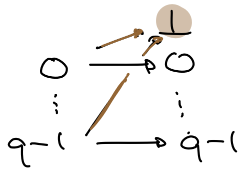
Alphabet: $\mathcal{A} = \mathbb{F}_q$ for q prime ← prime power ok, too

↳ $\{0,\dots,q-1\}$ with $+$ and $\cdot$ modulo q
(finite field with q elements)

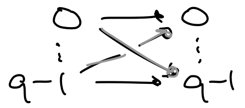
* Strange? No! e.g. with $q=257$ can encode 1 byte per symbol
* large q protects naturally against "burst errors"

Parameters: $K<N<q$ and $\alpha \in \mathbb{F}_q$

* overhead: $T := N-K$



* Can correct up to T erasures (= known error locations)
or up to $\frac{T}{2}$ errors at unknown locations



* α should be a "generator": $\mathbb{F}_q = \{0, \alpha, \alpha^2, \dots, \alpha^{q-1} = 1\}$ any nonzero element is power of α

always exists! e.g. $\mathbb{F}_3 = \{0, 2, 2^2 = 1\}$, $\mathbb{F}_5 = \{0, 2, 2^2 = 4, 2^3 = 3, 2^4 = 1\}$

all equalities today are "modulo q"

↳ generator polynomial: $G = (Z - \alpha) \cdots (Z - \alpha^T)$

variable of the polynomial

Encoder: Input: $s^K \in \mathcal{A}^K$

* $P \leftarrow s_1 + s_2 Z + \dots + s_K Z^{K-1}$
* $R \leftarrow P \cdot Z^T \bmod G$ ← degree $< T$ (= degree of G)

remainder of poly division (see ex. class)

* $M \leftarrow P \cdot Z^T - R$ ← degree $N-1$ & leading coeffs $s_K, \dots, s_1$
* $x^N \leftarrow$ coefficients of M ← i.e. $M = x_1 + x_2 Z + \dots + x_N Z^{N-1}$

By construction:

* $x^N = [x_1, \dots, x_T, \overbrace{s_1, \dots, s_K}^{\text{source message}}]$ M and $P \cdot Z^T$ differ in degree $< T$ only!

* M is multiple of G we subtracted the remainder!

$\Rightarrow$ "parity checks" $\boxed{M(\alpha) = \ldots = M(\alpha^T) = 0}$ $\circledast$

$\boxed{\text{ex:}}$ $K=1$, $N=3$, $q=5$ and $\alpha=2$

↳ $T=2$ & $G=(Z-2)(Z-4) = Z^2 - Z - 2$ (mod 5!)

To encode $s \in \mathbb{F}_5$:

* $P \leftarrow s$
* $R \leftarrow s \cdot Z^2 \text{ mod } G = s\cdot Z^2 - s \cdot G = s \cdot Z + 2s$
* $M \leftarrow s \cdot Z^2 - R = s \cdot Z^2 - s \cdot Z - 2s$
* $x^N \leftarrow [-2s, -s, s]$ ⤳ linear code!

$\uparrow$ as claimed above

$\boxed{\text{How to decode?}}$ Imagine we receive $y^N \in \mathcal{A}^N$.

Interpret as coeffs of polynomial:

$$R = M + E$$

with error polynomial $E = \sum_{k=1}^{c} e_k Z^{i_k}$

#errors

locations $\in \{0, \ldots, N-1\}$
$\uparrow \quad \uparrow$
grr...

mismatch

Two settings:

* Erasures: e_k unknown, c and i_k known

* General errors: everything unknown

What do we know? $\circledast$ implies:

$$\text{(I)} \left\{ \begin{aligned} E(\alpha) &= \sum_{k=1}^{c} e_k \alpha^{i_k} = R(\alpha) \\ &\vdots \\ E(\alpha^T) &= \sum_{k=1}^{c} e_k \alpha^{T \cdot i_k} = R(\alpha^T) \end{aligned} \right\}$$

T linear equations in unknowns $e_1, \ldots, e_c$

... if locations $i_1, \ldots, i_c$ known

This solves the problem for <u>erasure errors</u>: Can correct $\boxed{c \leq T \text{ erasures}}$

<u>ex:</u> $x^N = [-\cancel{2s}, -s, \cancel{s}]$

imagine $T=2$ erasure errors, e.g. $y^N = [0, -s, 0]$.

Known locations

$$R = -sZ \qquad E = e_1 Z^0 + e_2 Z^2 = e_1 + e_2 Z^2$$

$$\begin{aligned} E(2) &= e_1 + e_2 4 \overset{!}{=} R(2) = -2s \\ E(4) &= e_1 + e_2 \overset{!}{=} R(4) = s \end{aligned} \Rightarrow e_1 = 2s,\ e_2 = -s,\ E = 2s - sZ^2$$

$$\Rightarrow M = R - E = -2s - sZ + sZ^2 \,\hat{=}\, [-2s, -s, s]$$

☺

Decoder for erasures: <u>Input</u>: $y^N \in \mathcal{A}^N$, error locations $i_1, \dots, i_c$

- $R \leftarrow y_1 + y_2 Z + \dots + y_N Z^{N-1}$
- Solve ① for $e_1, \dots, e_c$
- $E \leftarrow e_1 Z^{i_1} + \dots + e_c Z^{i_c}$
- $M \leftarrow R - E$
- $\hat{S}^K \leftarrow$ leading K coeffs of M (i.e. $\hat{s}_1 = m_{N-K+1}, \dots, \hat{s}_K = m_N$)

What if locations unknown? Consider <u>locator polynomial</u>:

$$L := \prod_{k=1}^{c} (1 - Z\alpha^{i_k}) = 1 + L_1 Z + \dots + L_c Z^c$$

Should all be distinct: need $N \leq q-1$

Roots are α^{-i_k} for $k = 1, \dots, c$. How to determine L?

$$0 = \sum_k e_k \alpha^{i_k(j+c)} \underbrace{L(\alpha^{-i_k})}_{=0}$$

$$= E(\alpha^{j+c}) + L_1 \cdot E(\alpha^{j+c-1}) + \dots + L_c \cdot E(\alpha^j) \qquad \text{for } j = 1, 2, \dots$$

But: $E(\alpha) = R(\alpha), \dots, E(\alpha^T) = R(\alpha^T)$:

$$② \begin{bmatrix} R(\alpha^c) & \cdots & R(\alpha) \\ \vdots & & \vdots \\ R(\alpha^{2c-1}) & \cdots & R(\alpha^c) \end{bmatrix} \begin{bmatrix} L_1 \\ \vdots \\ L_c \end{bmatrix} = \begin{bmatrix} -R(\alpha^{c+1}) \\ \\ -R(\alpha^{2c}) \end{bmatrix} \quad \leftarrow \text{linear system for } L_1, \dots, L_c$$

... as long as $2c \le T$, i.e., $\boxed{c \le \frac{T}{2} \text{ errors}}$.

Still don't know c — so just try from $c = \lfloor \frac{T}{2} \rfloor, \dots, 1$ until ② unique solution.

Once we know L: search roots α^{-i_k} ⇝ i_k ⇝ e_k ⇝ E.

$\boxed{\text{ex:}}$ $S = 1$ is encoded in $x^N = [-2, -1, 1]$

Assume we receive $y^N = [-2, -1, 0]$ ⇝ $R = -2 - Z$

$$\left.\begin{array}{l} R(\alpha) = 1 \neq 0 \\ R(\alpha^2) = -1 \neq 0 \end{array}\right\} \Longrightarrow \text{error(s) happened.}$$

Try $\boxed{c = 1}$:

- Determine L: ②: $\overset{1}{R(\alpha)} \cdot L_1 = -\overset{-1}{R(\alpha^2)}$

  $\Longrightarrow L_1 = 1$, i.e. $L = 1 + Z$

- Determine error locations: L has root $\beta_1 = -1 = 4 = \alpha^2 = \alpha^{-2}$

  ↳ location $i_1 = 2$ → $E = eZ^2$

- Determine E and correct: ①: $E(\alpha) = 1$ $\Longrightarrow$ $e = -1$, $E = -Z^2$

  $E(\alpha^2) = -1$

  $\Longrightarrow M = R - E = -2 - Z + Z^2 \mathrel{\hat{=}} [-2, -1, 1]$

In general, have the following algorithm:

Decoder for general errors:

Input: $y^N \in \mathcal{A}^N$

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* $R \leftarrow y_1 + y_2 Z + \dots + y_N Z^{N-1}$
* If $R(\alpha) = \dots = R(\alpha^T) = 0$:
      $M \leftarrow R$
  else:
      For $C = \lfloor \frac{T}{2} \rfloor, \dots, 1$:
          If Det=0 in ②: Continue
          Solve ② for $L_1, \dots, L_C$
          $L \leftarrow 1 + L_1 Z + \dots + L_C Z^C$
          $\rho_1, \dots, \rho_C \leftarrow$ roots of $L$                    ← — search
          For k=1,...,C:
              $i_k \leftarrow$ number in $\{0, \dots, N-1\}$ s.th. $\rho_k = \alpha^{-i_k}$   ← — search/look up
                                                          $= \alpha^{q-1-i_k}$
          Solve ① for $e_1, \dots, e_C$
          $E \leftarrow \sum_{k=1}^{C} e_k Z^{i_k}$
          $M \leftarrow R - E$
          Break
* $\hat{s}^K \leftarrow$ leading $K$ coeffs of $M$ (i.e. $\hat{s}_1 = m_{N-k+1}, \dots, \hat{s}_k = m_N$)
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Appendix: Why does ① have a unique solution if $\boxed{c \le T}$?

We use linear algebra, which works the same over $\mathbb{F}_q$ as over $\mathbb{R}$ or $\mathbb{C}$.

Consider the following $T \times T$-matrix, where $\beta_1, \dots, \beta_T$ are arbitrary:

$$B = \begin{bmatrix} \beta_1 & \cdots & \beta_T \\ \vdots & & \\ \beta_1^T & & \beta_T^T \end{bmatrix}$$

- * $\det(B)$ is polynomial of degree $1+2+\dots+T = \frac{T(T+1)}{2}$ in $\beta_1, \dots, \beta_T$
- * $\det(B) = 0$ if $\beta_i = 0 \implies \beta_i \overset{\text{divides}}{\mid} \det B$
- * $\det(B) = 0$ if $\beta_i = \beta_j \implies \beta_i - \beta_j \overset{\text{divides}}{\mid} \det B$

$\implies \det(B)$ proportional $\beta_1 \cdots \beta_T \prod_{i<j} (\beta_i - \beta_j)$ (same degree!)

RESULT: $\boxed{\text{If } \beta_1, \dots, \beta_T \text{ distinct and nonzero then } \underbrace{B \text{ is invertible}}_{\text{In particular: all columns linearly independent!}}}$

Now note that our linear system ① is of the following form:

$$\underbrace{\begin{bmatrix} \alpha^{i_1} & \cdots & \alpha^{i_c} \\ \vdots & & \vdots \\ (\alpha^{i_1})^T & \cdots & (\alpha^{i_c})^T \end{bmatrix}}_{} \begin{bmatrix} e_1 \\ \vdots \\ e_c \end{bmatrix} \overset{!}{=} \begin{bmatrix} R(\alpha) \\ \\ \\ R(\alpha^T) \end{bmatrix} \qquad (c \le T)$$

linearly independent columns, since $\beta_k = \alpha^{i_k}$ distinct and nonzero!

indeed: since α is "generator",

$$\mathbb{F}_q = \{0, \underbrace{\alpha, \dots, \alpha^{q-1} = 1}_{\text{all distinct}}\}$$

and $0 \le i_1 \ne \dots \ne i_c \le N-1 < q-1$

THUS: $\boxed{\text{linear system has unique solution}}$

(solution exists by assumption)