Introduction to Information Theory, Fall 2021

Homework problem set #4   **due Nov 28, 2021**

> **Rules:** Always explain your solutions carefully. Please hand in the assignment in groups on Canvas. In the werkcollege the TAs can tell you more about how this works.

1. **Arithmetic coding (1 point):** Consider the following language model for producing strings of length $N = 2$:

x_1	$P(x_1)$
A	$5/6$
H	$1/6$

$P(x_2\|x_1)$	$x_1 = $ A	$x_1 = $ H
$x_2 = $ i	$1/3$	$1/5$
$x_2 = $ y	$2/3$	$4/5$

   For example, the probability of the string 'Hi' according to this model would be $1/6 \cdot 1/5 = 1/30$. We sort the alphabet as follows: A $<$ H and i $<$ y.

   (a) Encode the string Hi using the arithmetic coding algorithm. Visualize all relevant intervals and indicate at which point the algorithm outputs which bits.

   (b) The bitstring 10 was produced by encoding a string $x_1 x_2$ with the arithmetic coding algorithm. Determine $x_1 x_2$.

2. **Symmetric channels (1 point):** Any channel $Q(y|x)$ can be seen as a matrix whose rows are indexed by y and whose columns are indexed by x. This matrix is called the *transition matrix*. A channel is called *weakly symmetric* if the columns of the transition matrix are permutations of each other and if all row sums are equal. In this case the capacity is easy to calculate:

$$C(Q) = \log \#\mathcal{A}_Y - H(c), \tag{1}$$

   where c is an arbitrary column of the transition matrix.

   (a) Write down the transition matrix for the *binary symmetric channel* from class, confirm that the channel is weakly symmetric, and use (1) to compute its capacity.

   (b) Prove that the formula (1) holds for all weakly symmetric channels.

   Hint: Imitate the strategy discussed in class. Start with $I(X : Y) = H(Y) - H(Y|X)$, then compute $H(Y|X)$, and finally show that the uniform input distribution maximizes $H(Y)$.

3. ▦ **Arithmetic coding (1 point):**
   In this problem, you will implement the arithmetic coding algorithm discussed in class and compare its performance using different probabilistic models. To get started, open the notebook at https://colab.research.google.com/github/amsqi/iit21-homework/blob/master/04-homework.ipynb and follow the instructions.

   Please submit both the notebook **and** a PDF printout, or provide a link to your solution on Colab. You can achieve the maximum score if your solution produces the correct output. We will only have a closer look at your code in case of problems.