Probability Theory Refresher (§2)

Will be slightly informal (but in a way that can be made completely rigorous)
Axiomatic approach → text book / after class. When in doubt: ASK!

Probability distribution on $\mathcal{A}$ (finite set): $P: \mathcal{A} \to \mathbb{R}_{\geq 0}$, $\sum_{a \in \mathcal{A}} P(a) = 1$

e.g. Bernoulli(f): $\mathcal{A} = \{0,1\}$, $P(1) = f$, $P(0) = 1-f$

Uniform($\mathcal{A}$): $P(a) = \frac{1}{\# \mathcal{A}}$ $\forall a \in \mathcal{A}$

Random variable (RV) $X \triangleq$ prob. dist. P_X on set $\mathcal{A}_X$

NOTATION: $X \sim P$ for $P_X = P$

UNLIKE THE BOOK, I ALWAYS DISTINGUISH X and x

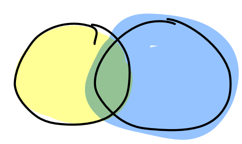
$\Pr(\underset{\text{RV}}{X} = \underset{\text{outcome}}{x}) = P_X(x) = P(x)$ we leave out subscript if clear!

$\Pr(X \in S) = \sum_{x \in S} P(x)$

$\Pr(\text{condition on } X) = \sum_{x \text{ cond. holds}} P(x) = \Pr(X \in \{x \text{ s.th. condition holds}\})$

e.g. if X random variable on $\{1, \ldots, 6\}$:
$\Pr(X \text{ even and } X \neq 2) = \Pr(X \in \{4,6\}) = P(4) + P(6)$

* $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B) - \underbrace{\Pr(A \text{ and } B)}_{= 0 \text{ if mutually exclusive}}$
  $\leq \Pr(A) + \Pr(B)$
  "union bound"



* X RV, f function $\Rightarrow Y = f(X)$ RV

$\Pr(Y = y) = \sum_{x: f(x) = y} \Pr(X = x)$ or simply $P(y) = \sum_{f(x) = y} P(x)$

More than one random variable

How to describe "pair of RVs" (X, Y)? "Joint prob. dist":

$\Pr(X = x, Y = y) = P_{(X,Y)}(x,y) = P_{XY}(x,y) = P(x,y)$

i.e. (X,Y) is RV on $\mathcal{A}_{XY} = \mathcal{A}_X \times \mathcal{A}_Y$. Similar for tuples.

* Can visualize by "probability table" or "contingency table":

$y \backslash x$	SUMMER	WINTER	
SUN	30%	10%	40%
RAIN	20%	40%	60%
	50%	50%	

* marginal distributions of X & Y:

$$P(x) = \sum_y P(x,y) \quad \& \quad P(y) = \sum_x P(x,y)$$

i.e. $Pr(X=x) = \sum_y Pr(X=x, Y=y)$ etc.

* X, Y are called independent if $P(x,y) = P(x)\cdot P(y)$

NOT independent! $p(\text{SUN},\text{SUMMER}) \neq p(\text{SUN})\cdot p(\text{SUMMER})$

How about:

15%	60%
5%	20%

independent!

Conditional prob. dist. of Y given X:

$$Pr(Y=y \mid X=x) := \frac{Pr(X=x, Y=y)}{Pr(X=x)}$$

NOTATION: $P_{Y|X=x}(y)$, $P_{Y|X}(y|x)$, $p(y|x)$, ...

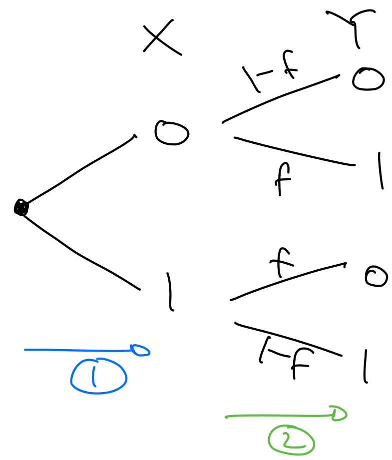
i.e. $P(y|x) = \dfrac{P(x,y)}{P(x)}$ and $P(x|y) = \dfrac{P(x,y)}{P(y)}$

* $P(y|x)$ is prob. dist in y for each fixed x!

Two simple rewritings:

* $P(x,y) = \underbrace{P(x)}_{1}\underbrace{P(y|x)}_{2} = P(y)P(x|y)$

e.g. X channel input, Y channel output, $P(y|x)$ channel



FORWARD

* Bayes rule:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)} = \frac{P(y|x)P(x)}{\sum_{x'} P(y|x')P(x')}$$

INVERSE

e.g. $P(\text{pos} \mid \text{sick}) = P(\text{neg} \mid \text{healthy}) = 90\%$, $P(\text{sick}) = 1\%$

$$\Rightarrow P(\text{sick} \mid \text{pos}) = \frac{0.9 \times 0.01}{0.9\times 0.01 + 0.1 \times 0.99} = \frac{1}{12} < 10\% \;!$$

e.g. decoding the repetition code R_3: assume $S \sim$ Uniform$(\{0,1\})$
all independent RV's

$Y_1 = S \oplus N_1, \ldots, Y_3 = S \oplus N_3$ where $N_1, N_2, N_3 \sim$ Bernoulli(f)

Sum modulo two (XOR)

Assume we received $y = y_1 y_2 y_3$. How should we estimate s?

$$P(s|y) = \frac{P(y|s)\overbrace{P(s)}^{=\frac{1}{2}}}{\underbrace{P(y)}_{\text{fixed}}}$$

$$\Rightarrow \frac{P(S=0|y)}{P(S=1|y)} = \frac{P(y|S=0)}{P(y|S=1)} = \frac{P(y|X=000)}{P(y|X=111)} = \prod_{k=1}^{3} \underbrace{\frac{P(y_k|X_k=0)}{P(y_k|X_k=1)}}_{\frac{1-f}{f} \text{ if } y_k=0, \text{ else } \frac{f}{1-f}}$$

$$= \underbrace{\left(\frac{1-f}{f}\right)}_{>1 \text{ if } f<50\%}^{\#0\text{'s} - \#1\text{'s}} = \begin{cases} >1 & \text{if } \#0\text{'s} > \#1\text{'s} \\ <1 & \text{if } \#1\text{'s} > \#0\text{'s} \end{cases}$$

= majority vote

Combining independent RV's: independent and identical distribution

$\boxed{\text{QUIZ:}}$ ① Let $X, N \overset{\text{IID}}{\sim}$ Uniform$(\{0,1\})$, $Y = X \oplus \underbrace{N}_{\text{uniform!}}$.
Are X and Y independent?

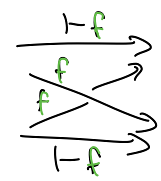
$\boxed{\text{YES!}}$ $\Pr(X=x, Y=y)$
$= \Pr(X=x, N=x\oplus y)$
$= \frac{1}{4} = \Pr(X=x)\Pr(Y=y)$

② How to label two dice w/ numbers from $0, \ldots, 6$
such that their sum is $\sim$ Uniform$(\{1,2,\ldots,12\})$?

A: 123456
B: 000666

Binomial(n, f): Distribution of $Y = X_1 + \ldots + X_n$ where $X_i \overset{\text{IID}}{\sim}$ Bernoulli(f)

* e.g. number of bit flips when we send n bits through



* $\Pr(Y=k) = \underbrace{\binom{n}{k}}_{\substack{\#\text{bitstrings} \\ \text{with } k \text{ ones} \\ n-k \text{ zeros}}} \underbrace{f^k (1-f)^{n-k}}_{\substack{\text{probability} \\ \text{of any such} \\ \text{string}}}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$ binomial coefficient

Next week: mean, variance, and their meaning + entropy + a 1st peek at compression.