Introduction to Information Theory (§1)

Mackay

① How to measure information? How to ask the most informative questions?

"bit"... but: 

→ "entropy"

"guess a number" game

→ data science, ML

② How to compress a data source? lossless: FLAC, ZIP, GIF, ... lossy: JPG, MP3, MP4, ...

③ How to reliably send information over unreliable channels? LTE, Blu-ray, QR-codes, ...

1948: Shannon, "A Mathematical Theory of Information" solved ①–③ "in theory"

origins: telecommunication + physics

Morse (1830s)

• E ••• S

1830s

1920s Bell labs

thermodynamics (1870+) Boltzmann, Gibbs, ...

info ~ log(#voltage levels) ~ log(#possible signals)

Nyquist   Hartley

abstraction!

today: engineering + theory (efficient codes, beyond i.i.d.) + quantum

Compression

Suppose we want to compress a message in $\{A,B,C,D\} = \mathcal{A}$:

source message $x \in \mathcal{A}$ → [Compressor] → code word $c \in \{0,1\}^\ell$ → [decompr.] → decompressed message $\hat{x} \in \mathcal{A}$

WANT: $\boxed{x = \hat{x}}$

4 possible messages $(2^2 = 4)$

→ need $\ell = 2$

x	c
A	00
B	01
C	10
D	11

why not

0

1

00

01

prefix

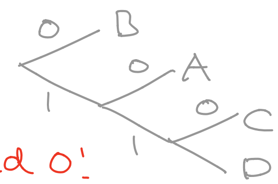
In general: $2^\ell \geq \#\mathcal{A} \implies \ell \geq \log_2(\#\mathcal{A})$

Can we do better? Imagine some messages are more frequent than others...

			code I	code II
A	sunshine	44%	10	10
B	rain	55%	0	0
C	snow	0.99%	110	11
D	hurricane	0.01%	111	0

longer   reused 0!

both can be decoded nicely! eg.



Code I: lossless, average length = 1.46 $\ll 2$ ⁉

Code II: lossy! $P_{error} = 0.01\%$, average length ≈ 1.45

How to do even better? Look at blocks of messages!

↳ $\boxed{\text{SHANNON:}}$ Optimal rate of compression is $\approx 1.06 \frac{\text{bits}}{\text{message}}$

entropy of source (but...)

$\boxed{\text{Communicating over Noisy Channels}}$

Examples of noisy channels & how to avoid:

* Scratch on Bluray disk — Don't do it!
* Loud party — Tell people not to shout!
* Mail arrives crumpled — Pay your postman more!
* Bad signal — Build more cell phone towers!
* Bit flip on hard disk — Shield better

$\Big\}$ € or infeasible

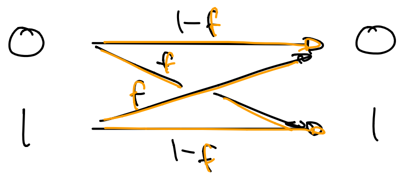
SATA mandates $P_{read\ error} \leq 10^{-14}$ ⇝ Reed-Solomon, LDPC codes

Mathematical model:

input → $\boxed{\text{Channel}}$ → output

$p(\text{output} \mid \text{input})$

e.g. binary symmetric channel:

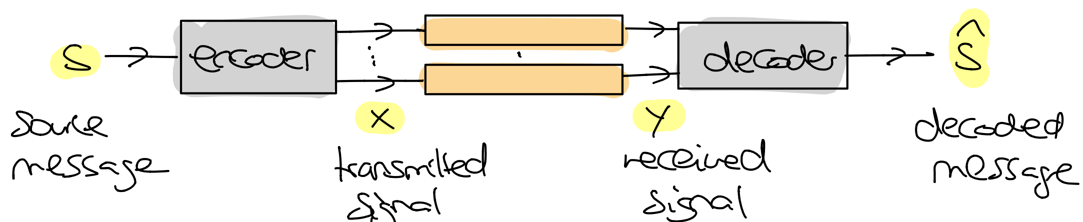


$$p(1|0) = p(0|1) = f$$
$$p(0|0) = p(1|1) = 1-f$$

f = probability of bit flip

assume we know f !!!

How to reduce error? Introduce redundancy by encoding message!



WANT: $\boxed{s = \hat{s}}$ with high probability!

Repetition code R_3:

* encoder:

s	$x = x_1x_2x_3$
0	000
1	111

* decoder: majority vote

$y = y_1y_2y_3$	$\hat{s}$
000	0
001 / 010 / 100	0
011 / 101 / 110	1
111	1

* analysis: Can deal with ≤ 1 bit flip

$\Longrightarrow P_{error} = Pr(2 \text{ or } 3 \text{ bit flips}) = \underbrace{3 \cdot f^2(1-f) + f^3}_{< f \text{ as long as } f < \frac{1}{2}} \approx 3f^2$ if f small

e.g. $f = 10\% = 0.1$: $\quad P_{error} = 0.028 \approx 0.03 = 3\%$ ☺

* $\text{rate} = \dfrac{\#\text{ source msg bits}}{\#\text{ transmitted bits}} = \dfrac{1}{3}$

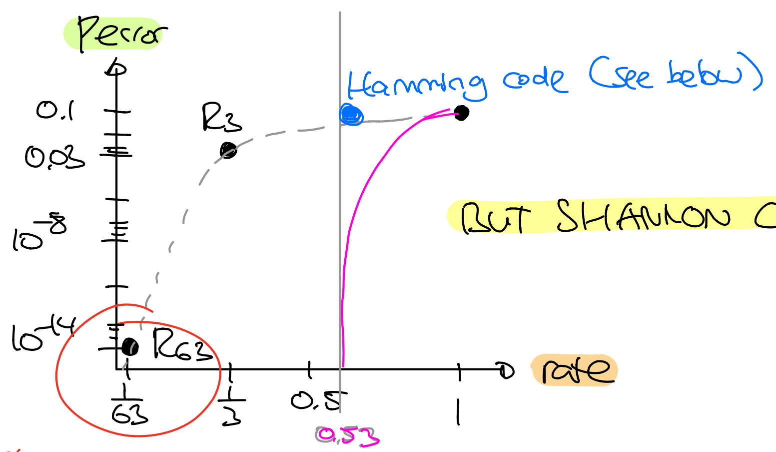
Ex: Show that this decoder is optimal (if $f \le 50\%$). Discuss $f = 50\%$.

What if we repeat $N > 3$ times?

$$P_{error} = Pr(\ge \tfrac{N}{2} \text{ bit flips}) \overset{\text{Thursday}}{=} \sum_{k \ge \frac{N}{2}} \binom{N}{k} f^k (1-f)^{N-k} \overset{\text{Later}}{\sim} 2^N f^{N/2} (1-f)^{N/2}$$

at $\text{rate} = \frac{1}{N}$

e.g. $f = 10\%$: $\quad P_{error} \sim 0.6^N$



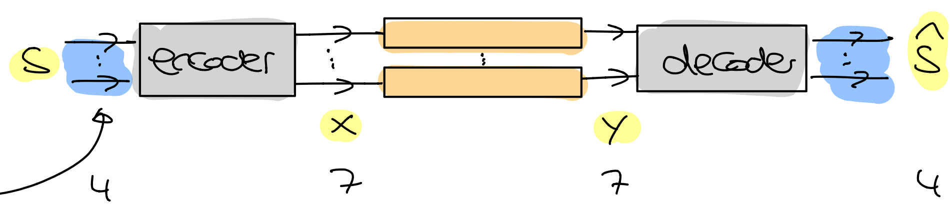
BUT SHANNON CAN DO BETTER! (see below)

it seems like rate → 0 if perror → 0

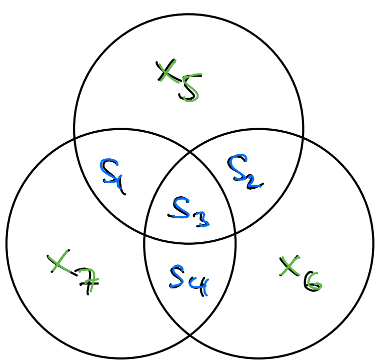
How can we find more & better codes?

Block codes:

Encode more than one symbol at a time



(7,4)-Hamming code:



$x_1 = s_1 \; \ldots \; x_4 = s_4$

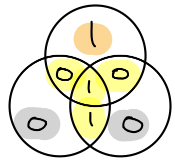
$x_5, \ldots, x_7$ chosen such that sum in each circle even

("parity bits")

$x_1 \cdots x_4$

$S = S_1 \dots S_4$	$x_5 x_6 x_7$
0000	000
0001	011
0010	111
0011	100
...	

e.g.



Any two codewords differ by 3 or more bits!

↳ can correct single bit flips

How to decode?

① Compute parities in all three circles: $z_1 = y_1 \oplus y_2 \oplus y_3 \oplus y_5 \pmod 2$

$\vdots$

z_3

② If at least one $z_i \neq 0$:

Flip unique bit that is <u>only</u> in circles with $z_i \neq 0$

$z = z_1 z_2 z_3$	000	001	010	100	011	101	110	111
flipped bit	—	y_7	y_6	y_5	y_4	y_1	y_2	y_3

$\Rightarrow$ $P_{\text{block error}} \leq \Pr(\geq 2 \text{ bit flips}) \sim \binom{7}{2} f^2 (1-f)^5 = 21 f^2$

$P_{\text{bit error}} = \frac{1}{4} \sum_{k=1}^{4} \Pr(\hat{s}_k \neq s_k) \sim 9 f^2$

exercise class

$\text{rate} = \frac{4}{7}$

SHANNON: For $f = 10\%$, can reliably send at optimal rate ≈ 0.53 !?

(but...)

Thursday: Probability theory recap + entropy (towards compression)