Tensors: From Entanglement to Computational Complexity

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arXiv:1709.0781, Proc. STOC'18

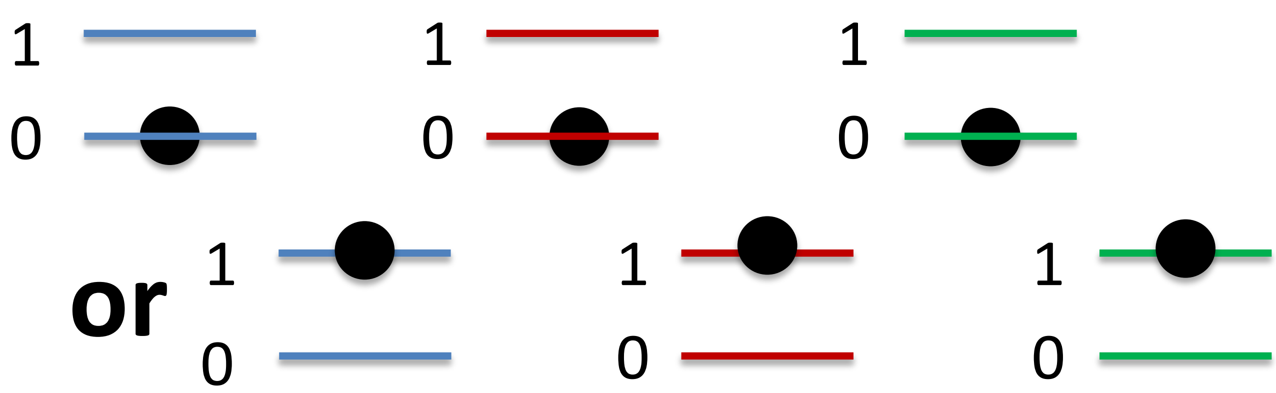
Outline

- Two motivations
- Resource theory of tensors
- Entanglement polytopes
- Tensor $\otimes$ tensor $\otimes$ …. $\otimes$ tensor
- Quantum functionals

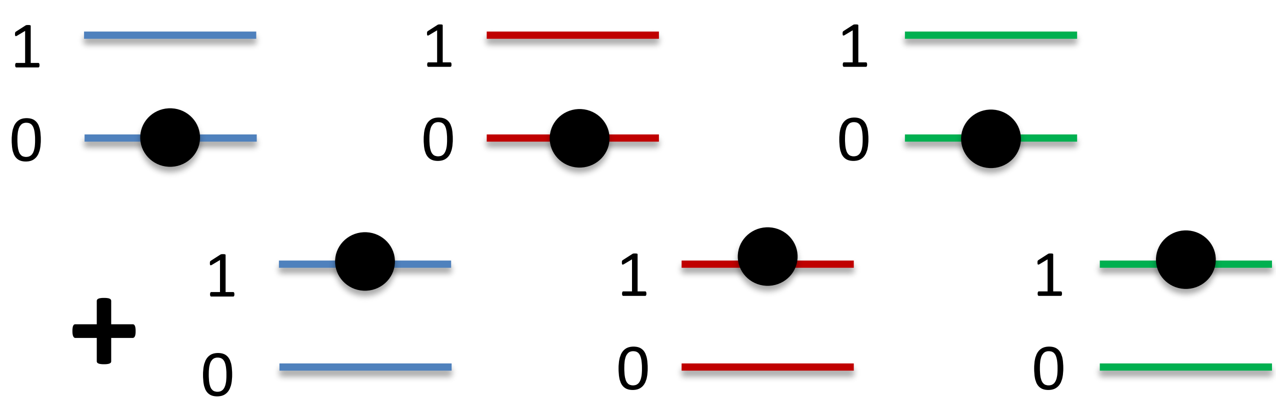
Two motivations

Quantum states

State of a classical system (3 bits)



State of a quantum system (3 qubits)



$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

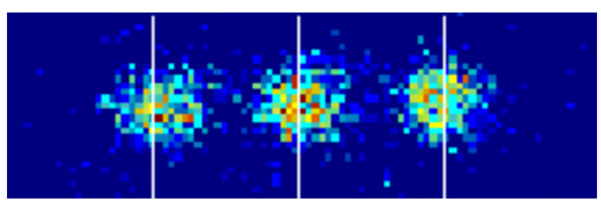
$$e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

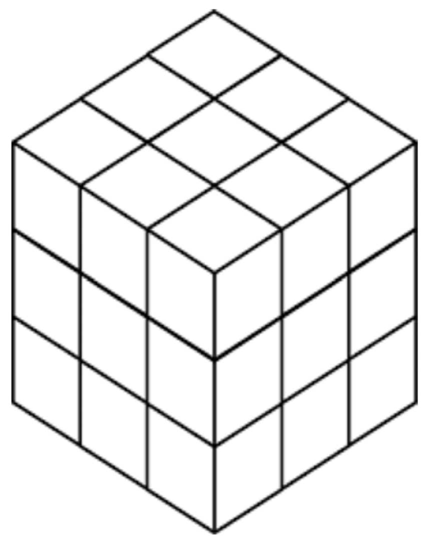
$$t = e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$$

Quantum state=tensor

$$t \in \mathbf{C}^d \otimes \mathbf{C}^d \otimes \mathbf{C}^d$$

$$t = \sum_{i,j,k=1}^{d} t_{ijk} e_i \otimes e_j \otimes e_k$$

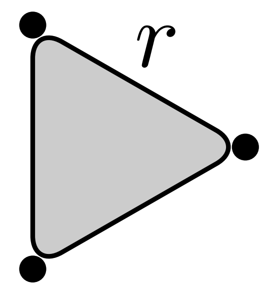


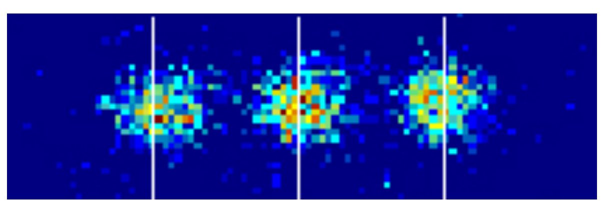


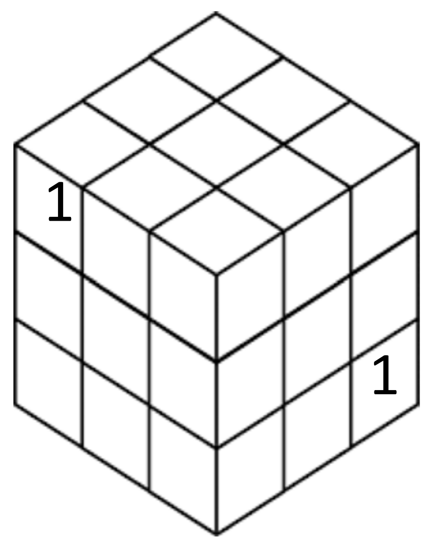
GHZ state = unit tensor

Greenberger-Horne-Zeilinger

$$\langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$

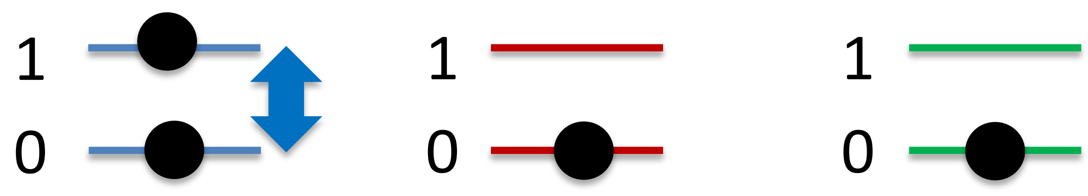




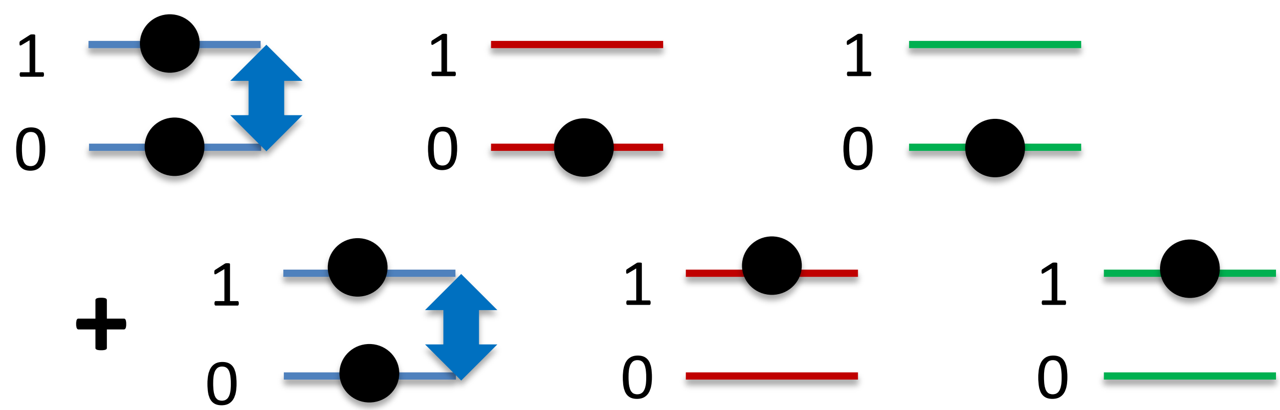


Local operations

Local transformation: Flip first bit



Local transformation: Flip first qubit



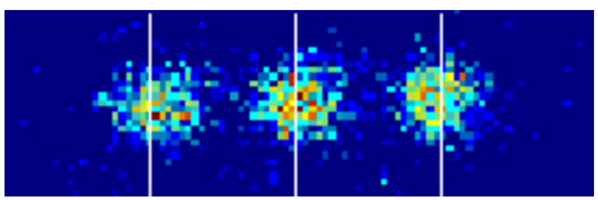
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

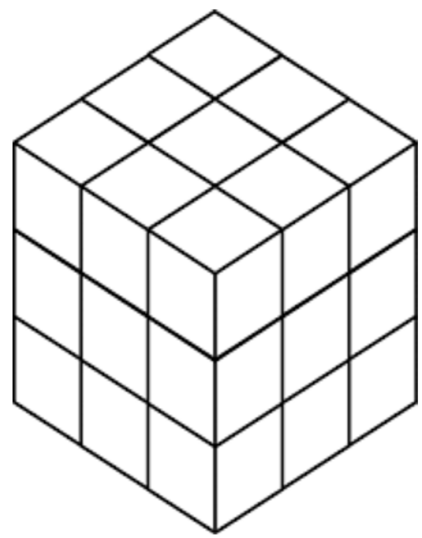
$$t = e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1$$

Local operations=restrictions

$$t \geq t' \text{ if } (a \otimes b \otimes c)\ t = t'$$

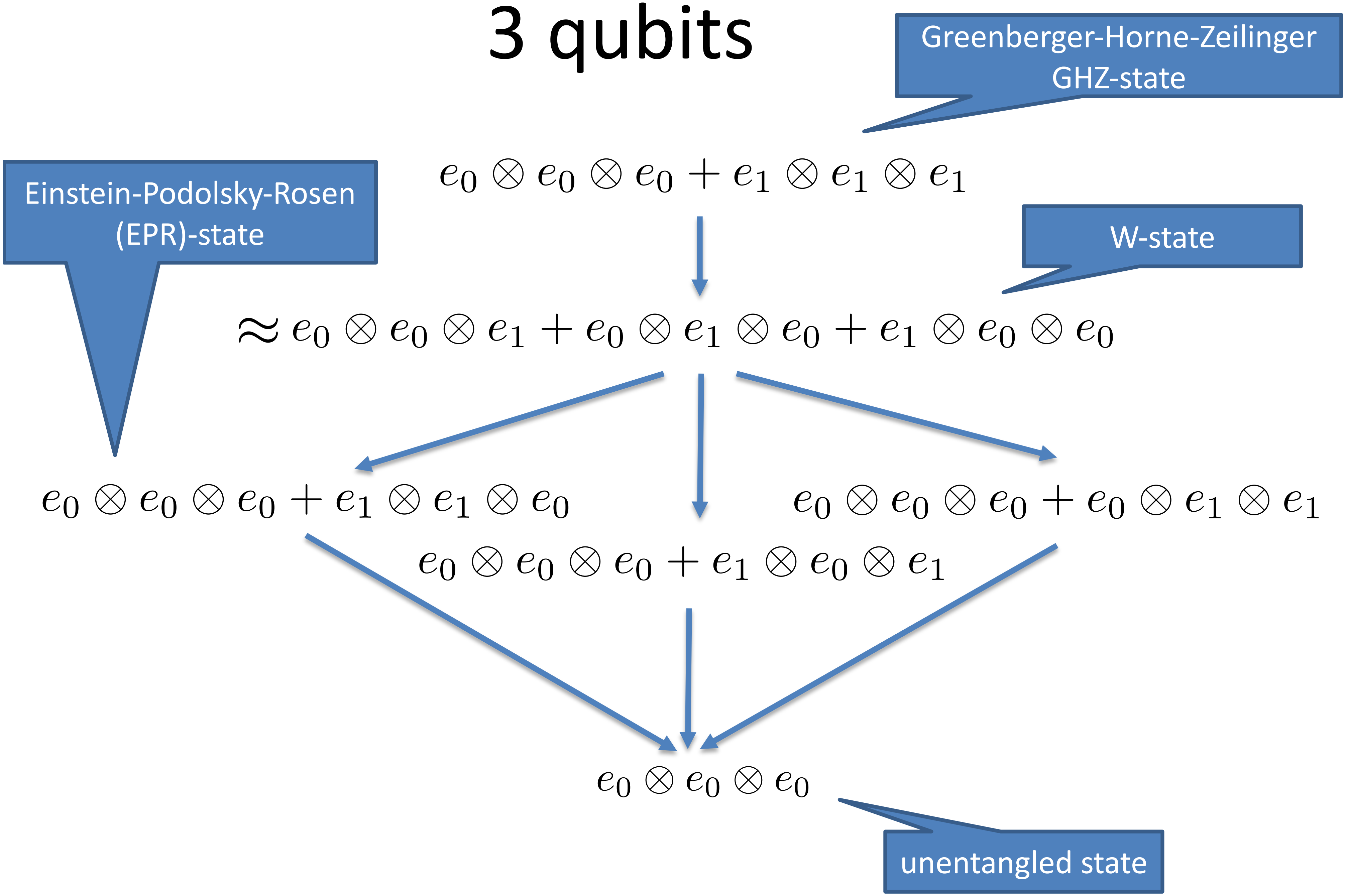
for some matrices a, b, c





Linear combination of slices

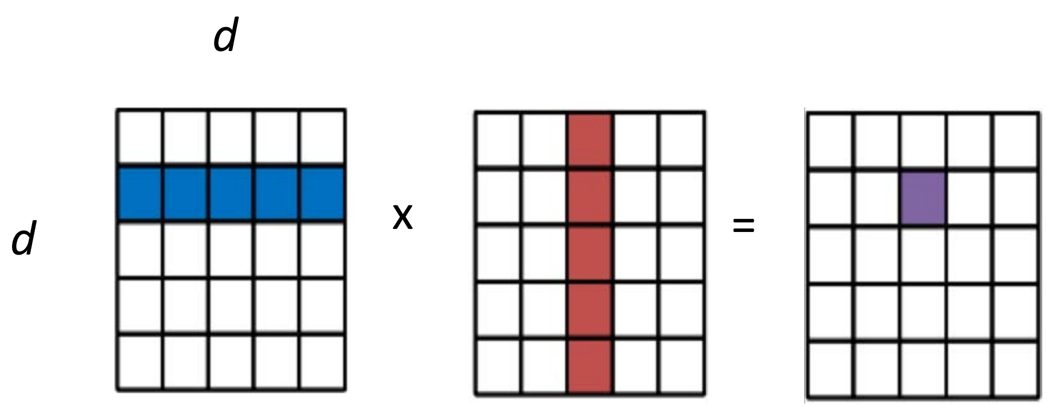
3 qubits


Greenberger-Horne-Zeilinger
GHZ-state
$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$
Einstein-Podolsky-Rosen
(EPR)-state
W-state
$\approx e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$
$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_0$
$e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1$
$e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_0 \otimes e_1$
$e_0 \otimes e_0 \otimes e_0$
unentangled state

Algebraic Complexity

$$M(d) = \text{ algebra of } d \times d \text{ complex matrices}$$

$$Mamu(d) : M(d) \times M(d) \to M(d) \qquad \text{bilinear}$$
$$(A, B) \mapsto A \cdot B$$



d^3 multiplications

Bilinear maps=tensors

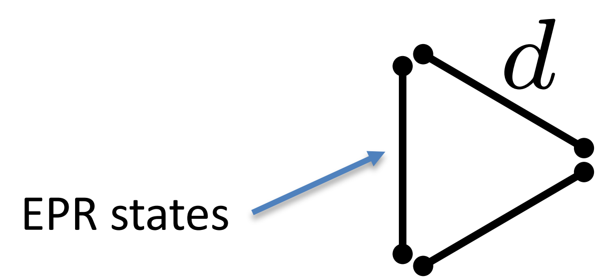
$$Mamu(d) : M(d) \times M(d) \times M(d)^* \to \mathbf{C}$$

$$(A, B, C) \mapsto tr A \cdot B \cdot C$$

$$Mamu(d) = \sum_{i,j,k=1}^{d} e_{ij} \otimes e_{jk} \otimes e_{ki}$$

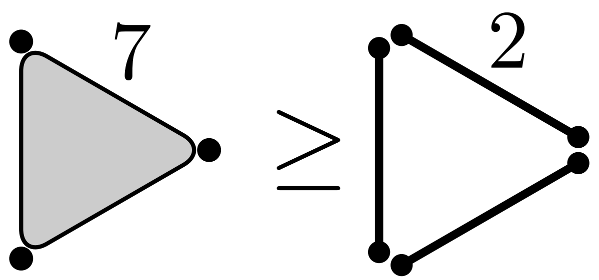
$$e_{ij} = e_i \otimes e_j$$

$$= \sum_{i,j,k=1}^{d} (e_i \otimes e_j) \otimes (e_j \otimes e_k) \otimes (e_k \otimes e_i)$$



Complexity=Tensor rank

Strassen: # elementary multiplications = tensor rank



Do you like Strassen's decomposition?
Then you might want to look at some tensor surgery next!
Ch. & Zuiddam,
Comp. Compl. 2018
arXiv:1606.04085

$$e_{00} \otimes e_{00} \otimes e_{00} + e_{11} \otimes e_{11} \otimes e_{11}$$
$$e_{01} \otimes e_{10} \otimes e_{00} + e_{10} \otimes e_{01} \otimes e_{11}$$
$$e_{01} \otimes e_{11} \otimes e_{10} + e_{10} \otimes e_{00} \otimes e_{01}$$
$$e_{00} \otimes e_{01} \otimes e_{10} + e_{11} \otimes e_{10} \otimes e_{01}$$

$$\begin{aligned} =& e_{-1} \otimes e_{1+} \otimes e_{00} + e_{1+} \otimes e_{00} \otimes e_{-1} + e_{00} \otimes e_{-1} \otimes e_{1+} \\ &- e_{-0} \otimes e_{0+} \otimes e_{11} - e_{0+} \otimes e_{11} \otimes e_{-0} - e_{11} \otimes e_{-0} \otimes e_{0+} \\ &+ (e_{00} + e_{11}) \otimes (e_{00} + e_{11}) \otimes (e_{00} + e_{11}) \end{aligned}$$

Resource theory of tensors

Resource theory of tensors

free operations → Restriction

valuable resource → t

- Restriction $\quad t \geq t' \text{ if } (a \otimes b \otimes c)\, t = t'$ for some matrices a, b, c
- Unit $\quad \langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$
- Rank $\quad R(t) = \min\{r : \langle r \rangle \geq t\}$
  $= \min\{r : t = \sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i\}$
- Subrank $\quad Q(t) = \max\{r : t \geq \langle r \rangle\}$

Restriction

$t \geq t'$ if $(a \otimes b \otimes c)\ t = t'$

for some matrices a, b, c

$t \cong t'$ if $t \geq t'$ and $t' \geq t$

iff $(a \otimes b \otimes c)\ t = t'$

for invertible a, b, c

iff $G.t = G.t'$ ← $G = GL(d) \times GL(d) \times GL(d)$

Deciding restriction ⟺ Classifying orbits and their relations

GHZ state

Degeneration

W state

$$(e_0 + \epsilon e_1)^{\otimes 3} - e_0^{\otimes 3}$$
$$= \epsilon(e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0) + O(\epsilon^2)$$

$$t \trianglerighteq t' \text{ if } t_\epsilon \underset{\epsilon \mapsto 0}{\longrightarrow} t', t \geq t_\epsilon$$

Deciding degeneration ⟺ Classifying orbit closures and their relations

Deciding degeneration

- Orbit closures are G-invariant algebraic varieties

$$t \ntrianglerighteq t' \text{ iff there exists}$$

$$G-\text{covariant polynomial } f:$$

$$f(t)=0, \text{ but } f(t') \neq 0$$

- Example: $e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$

$$\downarrow \quad \not\uparrow$$

f=Cayley hyperdeterminant

$$\approx \; e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$$

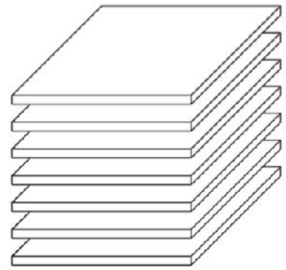
be happy with partial information

Entanglement polytopes

Local spectra

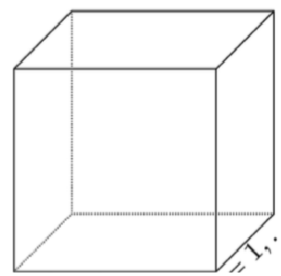
$$t'_A \in \mathbf{C}^d \otimes (\mathbf{C}^d \otimes \mathbf{C}^d)$$

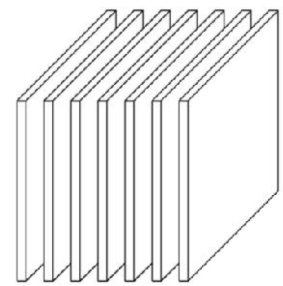
$$\lambda_A = \text{ singular values } (t'_A)^2$$



normalised

$$t' \in \mathbf{C}^d \otimes \mathbf{C}^d \otimes \mathbf{C}^d$$





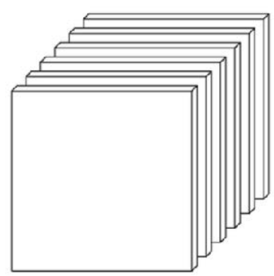
$$t'_B \in \ldots$$

$$\lambda_B = \text{ singular values } (t'_B)^2$$

ordered probability distribution
=spectrum of reduced density operator

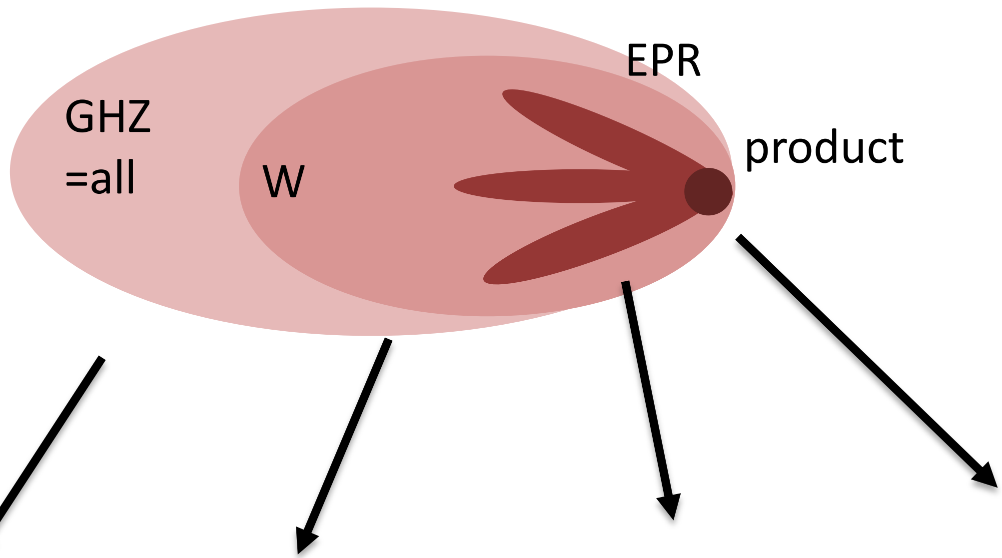
$$t'_C \in (\mathbf{C}^d \otimes \mathbf{C}^d) \otimes \mathbf{C}^d$$

$$\lambda_C = \text{ singular values } (t'_C)^2$$

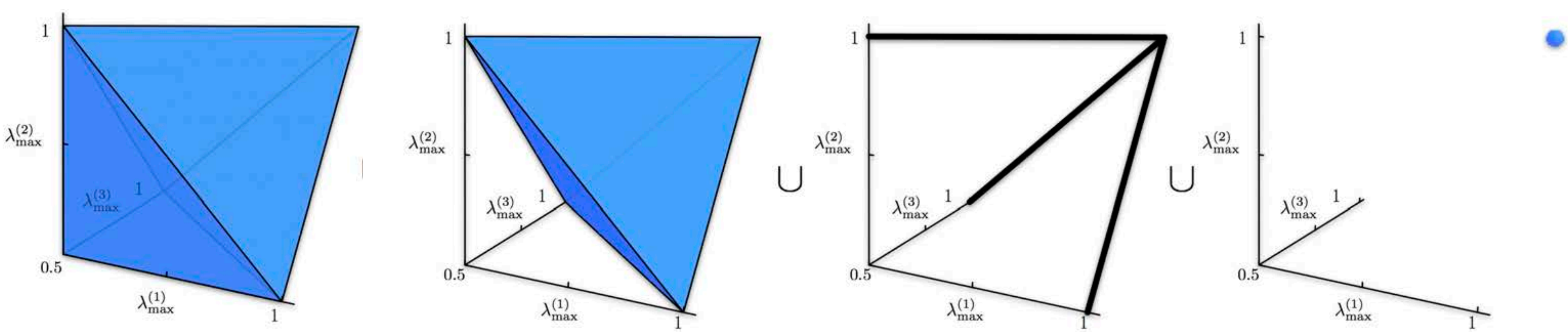


Entanglement polytopes

Reduced density matrices



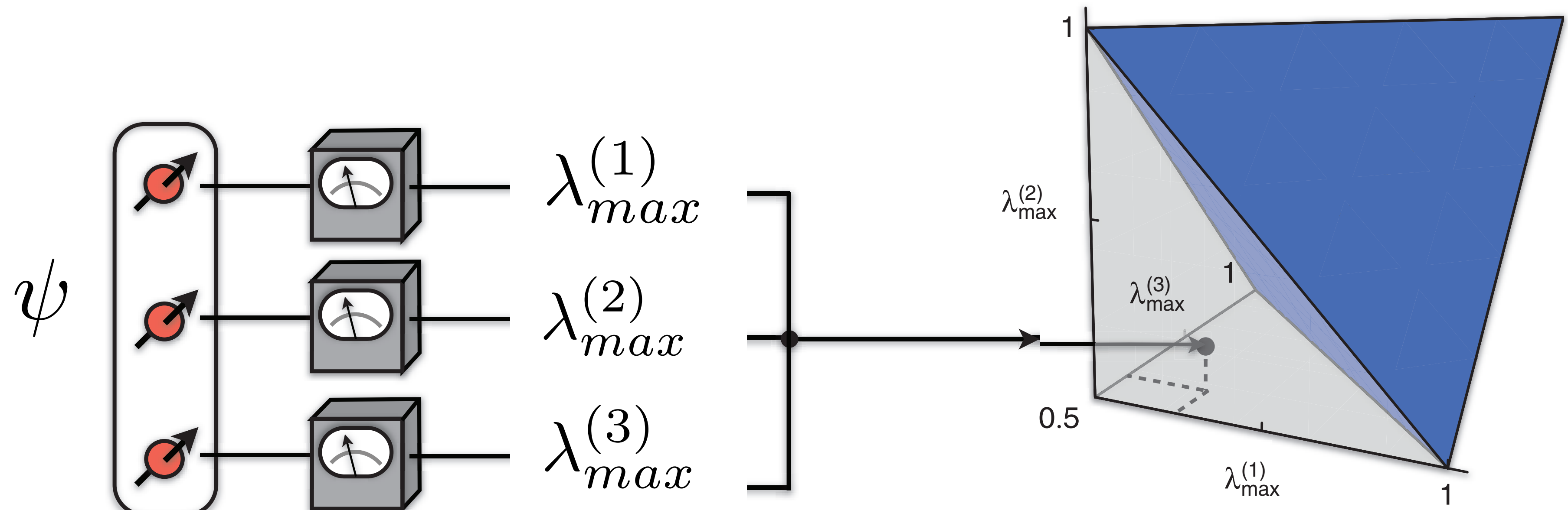
marginal
polytope



Ch-Mitchison, Klyachko,
Daftuar-Hayden (2004)
based in part on Kirwan

Walter-Doran-Gross-Ch,
Sawicki-Oszmaniec-Kus (2010) based on Brion

Experimental Detection



- if measured value
  - not in W-polytope
  - Then must be in GHZ-class!
- easy test for entanglement!

A little more partial information?

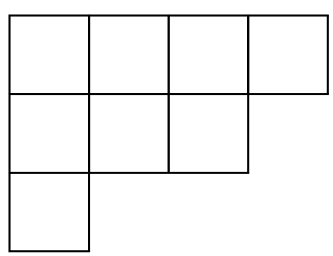
- Orbit closures are G-invariant algebraic varieties

$$t \ntrianglerighteq t' \text{ iff there exists}$$

$$G-\text{covariant polynomial } f : f(t) \neq f(t')$$

$$f(t) = 0, \text{ but } f(t') \neq 0$$

- f’s come in types indexed by 3 Young diagrams

$\lambda_A =$ 

boxes=degree

Weyl's construction

- Schur-Weyl duality

S_n acts $\quad$ $GL(d)$ acts

$$(\mathbf{C}^d)^{\otimes n} \cong \bigoplus_\lambda [\lambda] \otimes V_\lambda$$

- P_{λ_A} orthogonal projector onto λ_A component

$$\underbrace{(P_{\lambda_A} \otimes P_{\lambda_B} \otimes P_{\lambda_C})}_{=:P_\lambda} t^{\otimes n}$$

$$= \left(\sum_i v_i v_i^*\right) t^{\otimes n} = \sum_i v_i^* f_i(t)$$

Relaxation

- Orbit closures are G-invariant algebraic varieties

$$t \not\trianglerighteq t' \text{ iff there exists}$$

$$G-\text{covariant polynomial } f:$$

$$f(t)=0, \text{ but } f(t')\neq 0$$

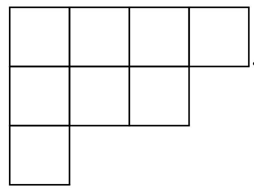
$$\text{if there is } \lambda \text{ s.th.}$$

$$P_\lambda t^{\otimes n}=0 \text{ but } P_\lambda t'^{\otimes n}\neq 0$$

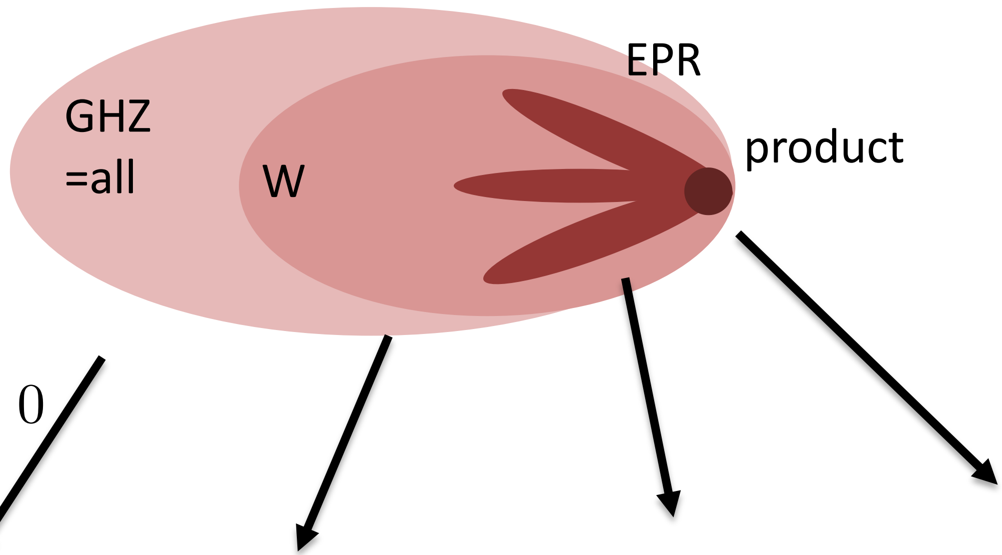
occurrence obstructions (Geometric Complexity Theory)
Mulmuley-Sohoni, Strassen, Bürgisser-Ikenmeyer, …

Entanglement polytopes

Invariant-theoretic



$$\longrightarrow \left(\frac{4}{8}, \frac{3}{8}, \frac{1}{8}\right)$$



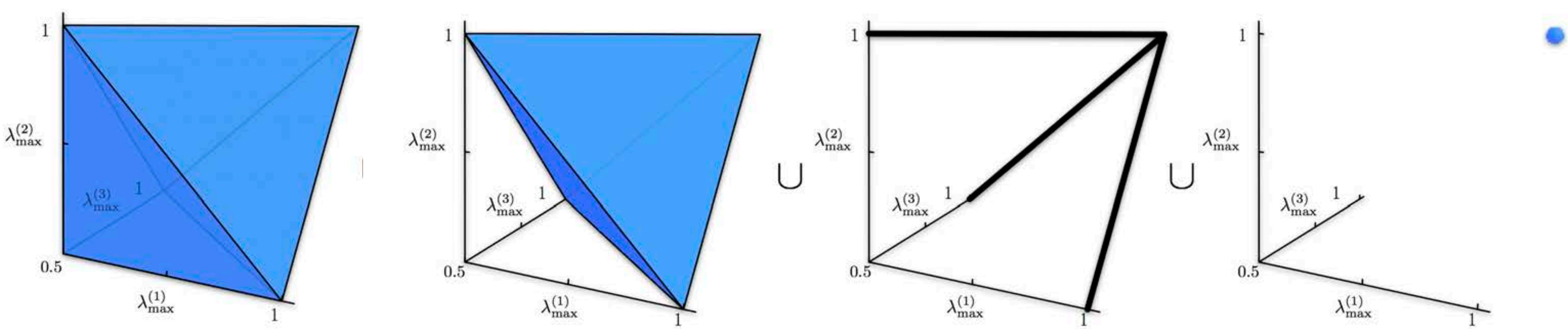
$g_\lambda \neq 0$

Kronecker

= marginal

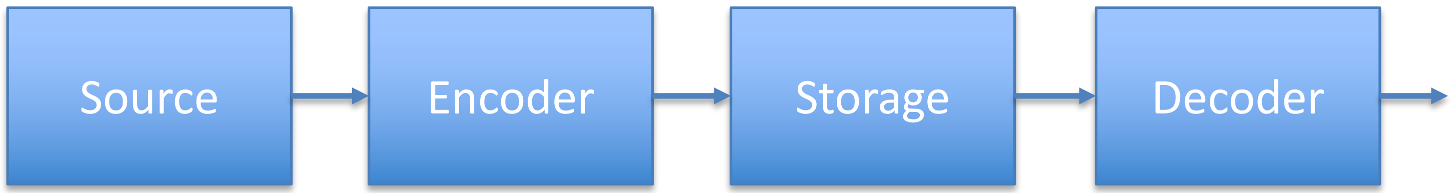
polytope

$P_\lambda t^{\otimes n} \neq 0$

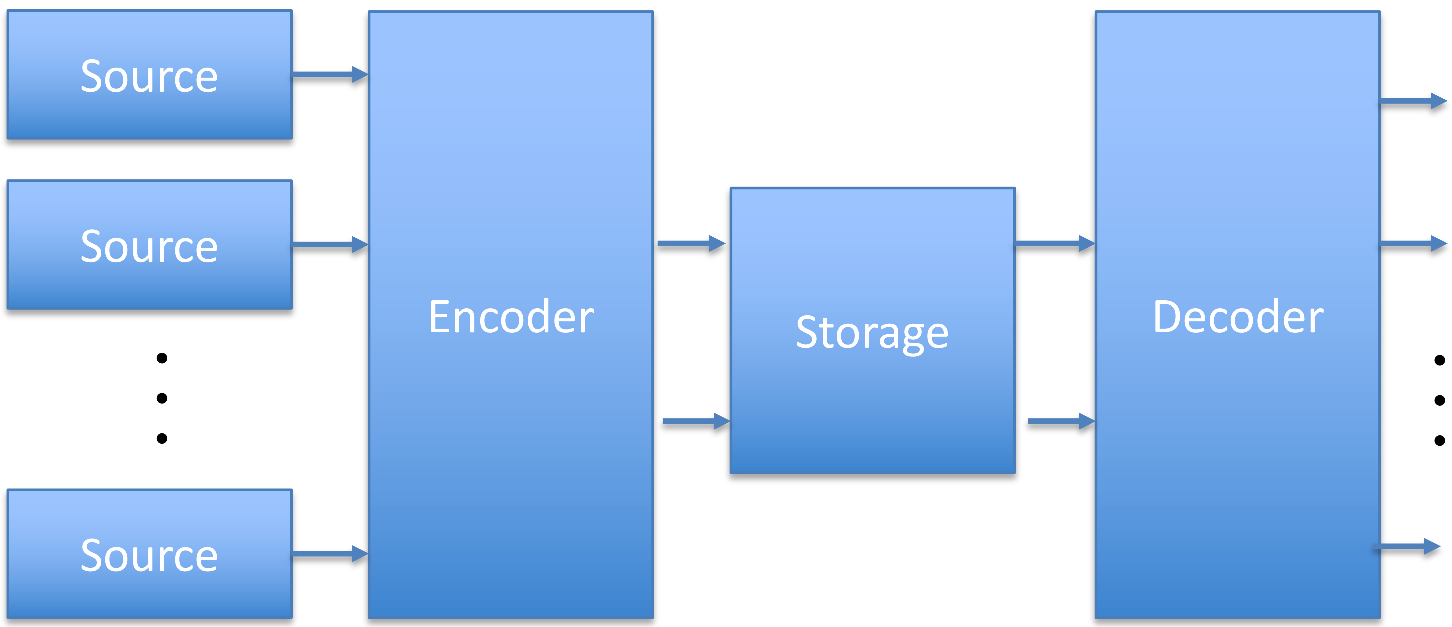


$$\text{tensor} \otimes \text{tensor} \otimes \ldots \otimes \text{tensor}$$

(Quantum) information theory



Shannon: storage cost= all bits



Shannon: storage cost= H(X) bits/symbol

A small observation

$$d = 2^n$$

$$e_i = e_{i_1 i_2 \cdots i_n} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$$

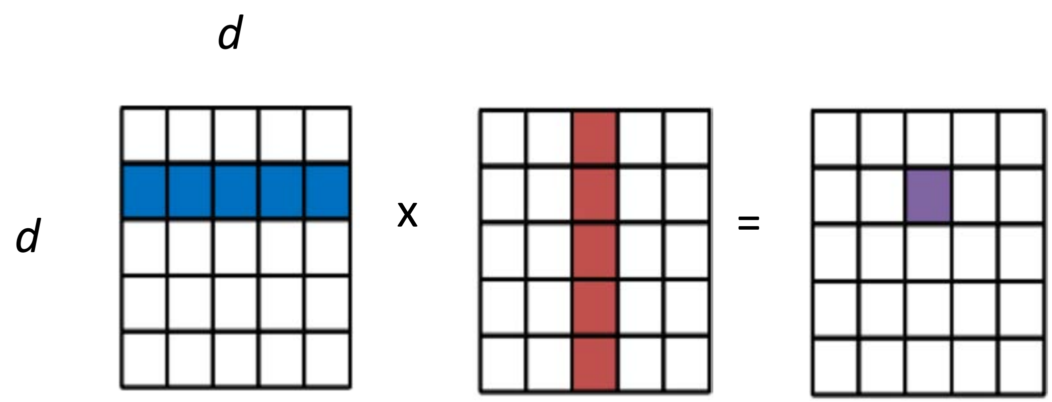
$$\sum_{i=1}^{d} e_i \otimes e_i = \left(\sum_{i_1=1}^{2} e_{i_1} \otimes e_{i_1} \right) \otimes \left(\sum_{i_2=1}^{2} e_{i_2} \otimes e_{i_2} \right) \otimes \cdots \otimes \left(\sum_{i_n=1}^{2} e_{i_n} \otimes e_{i_n} \right)$$

$$= (e_0 \otimes e_0 + e_1 \otimes e_1)^{\otimes n}$$

$$\langle d \rangle = \sum_{i=1}^{d} e_i \otimes e_i \otimes e_i = (e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1)^{\otimes n} = \langle 2 \rangle^{\otimes n}$$

$$Mamu(d) = \sum_{i,j,k=1}^{d} e_{ij} \otimes e_{jk} \otimes e_{ki} = \left(\sum_{i,j,k=1}^{2} e_{ij} \otimes e_{jk} \otimes e_{ki} \right)^{\otimes n} = Mamu(2)^{\otimes n}$$

Algebraic complexity theory



d^3 multiplications

- Exponent of matrix multiplication $O(d^\omega)$

$$2 \leq 2.38 \leq \cdots \leq 2.8 \leq 3$$

…, Coppersmith-Winograd Strassen

$$\omega = \inf\{r : \langle 2 \rangle^{\otimes(nr+o(n))} \geq Mamu(2)^{\otimes n}\}$$

- Conjecture: $\langle 2 \rangle^{\otimes 2n+o(n)} \geq Mamu(2)^{\otimes n}$

Asymptotic resource theory

- Asymp. restriction $t \gtrsim t' \text{ if } t^{\otimes n+o(n)} \geq t'^{\otimes n}$
- Unit $\langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$
- Asymp. rank $\tilde{R}(t) := \lim_{n\to\infty} R(t^{\otimes n})^{\frac{1}{n}}$
- Asymp. subrank $\tilde{Q}(t) := \lim_{n\to\infty} Q(t^{\otimes n})^{\frac{1}{n}}$

$\tilde{R}(Mamu(2)) = 2^{\omega}$

Asymptotic analogue of completeness of invariants for degeneration

Strassen's spectral theorem

$$t \gtrsim t' \text{ iff } F(t) \geq F(t') \text{ for all } F:$$

F monotone — under restriction $F(s) \geq F(s')$ for all $s \geq s'$

F normalised — $F(\langle r \rangle) = r$

F multiplicative — $F(s \otimes s') = F(s) \cdot F(s')$

F additive — $F(s \oplus s') = F(s) + F(s')$

$$\tilde{R}(t) = \max_F F(t)$$

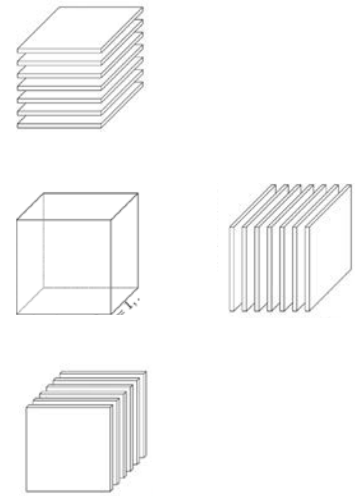
$$\tilde{Q}(t) = \min_F F(t)$$

$\Rightarrow$ easy

$\Leftarrow$ difficult

every *F* is an obstruction

What are the F's?



- Existence non-constructive
  - Compact space worth of them
  - 3 Gauge points: ranks of slicings
  - Construction of others open since '80s

- Theorem also true for subclasses of tensors
  - Oblique tensor
  - Strassen's support functionals

Main Result: Quantum functionals

$\theta = (\theta_A, \theta_B, \theta_C)$ probability distribution e.g. $\theta_A = \theta_B = \theta_C = \frac{1}{3}$

operator scaling

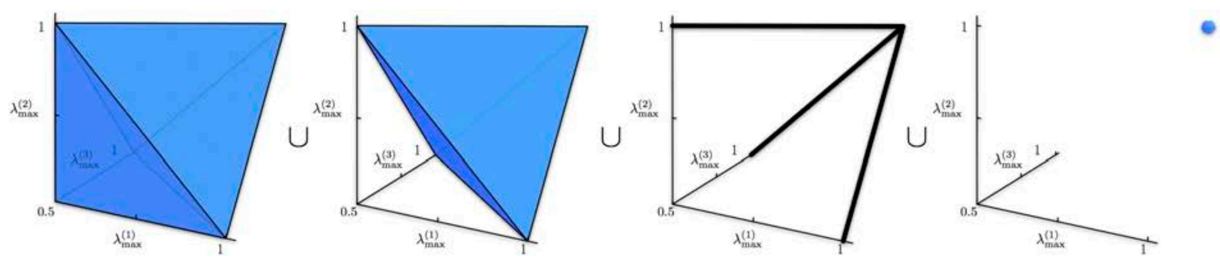
$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

entanglement polytope

$$F_\theta(t) := 2^{E_\theta(t)}$$

quantum functionals

Measures distance to origin (relative entropy distance)



$E_{(\frac{1}{3},\frac{1}{3},\frac{1}{3})}$	1	$h\left(\frac{1}{3}\right) \approx 0.92$	$\frac{2}{3}$	0

Main Result: Quantum functionals

$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

$$F_\theta(t) := 2^{E_\theta(t)}$$

F_θ monotone ← easy, since polytope gets smaller under restriction quantum functional gets smaller

F_θ normalised ← easy, since polytope of unit tensor contains uniform point $F(\langle r \rangle) = r$

F_θ multiplicative

F_θ additive ← similar to multiplicativity, see paper

Multiplicativity

$$F_\theta(t \otimes t') = F_\theta(t) \cdot F_\theta(t')$$

$$\Updownarrow \quad F_\theta(t) := 2^{E_\theta(t)}$$

$$E_\theta(t \otimes t') = E_\theta(t) + E_\theta(t')$$

$\geq$

Entanglement polytope:
Reduced density matrices

$\leq$

Entanglement polytopes:
Invariant-theoretic

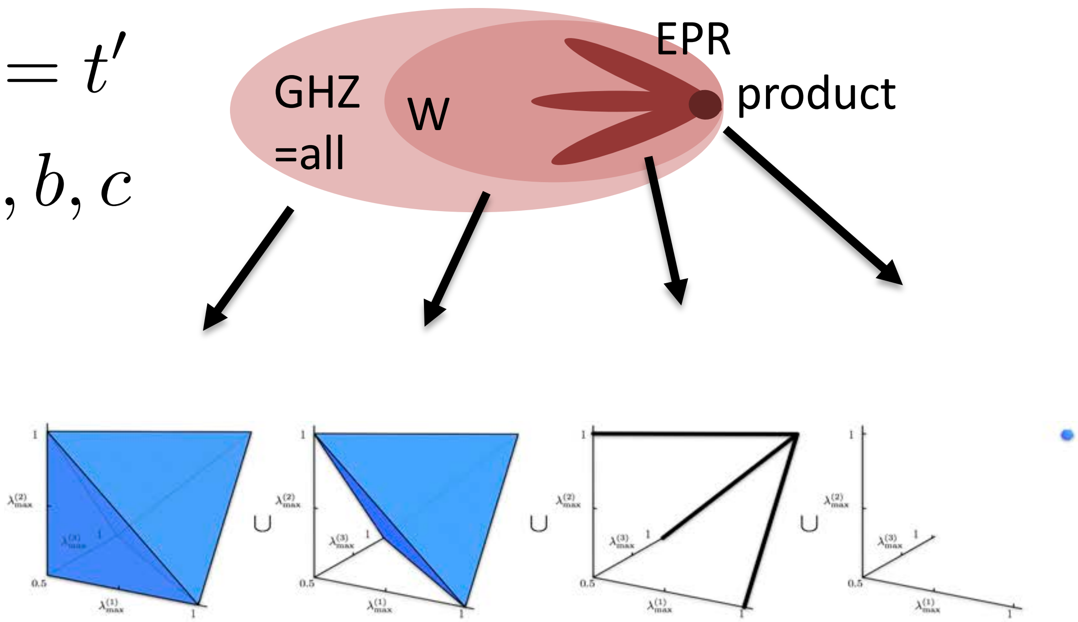
Quantum functionals: Some facts

- Extend Strassen's support functionals
- Are they complete?
  - If complete, then $\omega = 2$
- General setting of tensors of order k
- Connect Strassen's framework to capset
  - Reproves recent results
  - Characterise slice-rank

Summary

$t \geq t'$ if $(a \otimes b \otimes c)\ t = t'$

for some matrices a, b, c

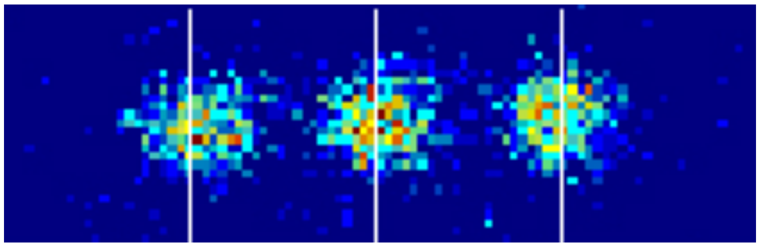


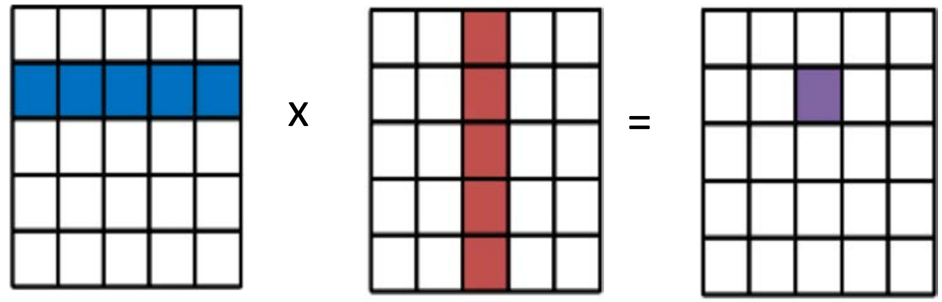
$t \gtrsim t'$ if $t^{\otimes n + o(n)} \geq t'^{\otimes n}$

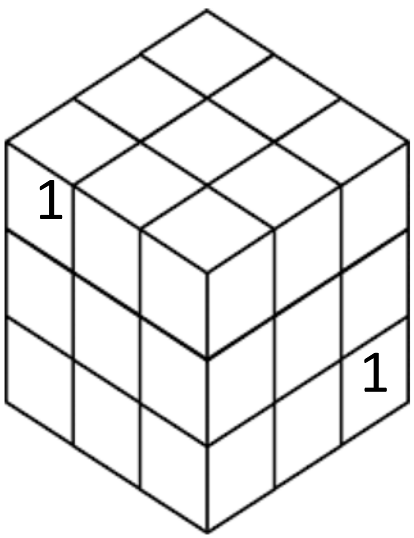
$$E_\theta(t) := \max_{\lambda \in \Delta(t)} \{\theta_A H(\lambda_A) + \theta_B H(\lambda_B) + \theta_C H(\lambda_C)\}$$

$$F_\theta(t) := 2^{E_\theta(t)}$$

If all, then $\omega = 2$




x
=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