The Paulsen problem, continuous operator scaling, and smoothed analysis

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Outline

Part I: Paulsen problem

- Motivation from frame theory

Part II: Continuous operator scaling

- Operator scaling, alternating algorithm, reduction
- Analysis of dynamical system

Part III: Smoothed analysis

- Proof outline, capacity lower bound

Part IV: Discussions

Frames

Frame: a collection of vectors $u_1, u_2, \dots, u_n \in \mathbb{R}^d$ that spans $\mathbb{R}^d$

Equal norm: if $\|u_i\|_2 = \|u_j\|_2$ for all i, j.

Parseval: if $\sum_{i=1}^{n} u_i u_i^T = I_d$.

An equal norm Parseval frame is an *overcomplete* basis:

$$\sum_{i=1}^{n} \langle x, u_i \rangle \; u_i = x \quad \forall\, x \in \mathbb{R}^d$$

It has applications in signal processing, communication theory, and quantum information theory.

Motivation

Equal norm Parseval frames are difficult to construct with only a few known algebraic constructions.

[Holmes-Paulsen 04] were interested in constructing Grassmaniann frames, equal norm Parseval frames with minimal $\max_{i,j} \langle u_i, u_j \rangle^2$, which are even more difficult to construct.

It is easier to construct "approximate" equal norm Parseval frames (e.g. random unit vectors, optimal packing of lines).

Question: Can we turn an "approximate" frame into an equal norm Parseval frame by just moving the vectors "slightly"?

The Paulsen Problem

What is the best function $f(n, d, \epsilon)$ such that for any $u_1, \ldots, u_n \in \mathbb{R}^d$ with

$$(1-\epsilon)\frac{d}{n} \le \|u_i\|_2^2 \le (1+\epsilon)\frac{d}{n} \quad \forall\, 1 \le i \le n \qquad (\epsilon - \textbf{nearly equal norm})$$

$$(1-\epsilon)I_d \preccurlyeq \sum_{i=1}^{n} u_i u_i^T \preccurlyeq (1+\epsilon)I_d \qquad (\epsilon - \textbf{nearly Parseval}),$$

there exist $v_1, \ldots, v_n \in \mathbb{R}^d$ with

$$\|v_i\|_2^2 = \frac{d}{n} \quad \forall\, 1 \le i \le n \quad \text{and} \quad \sum_{i=1}^{n} v_i v_i^T = I_d$$

such that

$$\sum_{i=1}^{n} \|u_i - v_i\|_2^2 \le f(n, d, \epsilon)?$$

Previous work

[Bodmann-Casazza, 10] $f(d,n,\epsilon) \leq O(d^{42}\, n^{18}\, \epsilon^{2})$ when $\mathbf{gcd}(d,n) = 1.$

- dynamical system improves on equal norm while keeping Parseval.

[Casazza-Fickus-Mixon, 12] $f(d,n,\epsilon) \leq O\left(d^{20/7}\, n^{2/7}\, \epsilon^{2/7}\right)$

- gradient descent improves on Parseval while keeping equal norm.

There are examples showing that $f(d,n,\epsilon) \geq d\epsilon$.

Question: Can the bound be independent of n?

Main Result

$$\boxed{\textbf{Theorem.}\quad f(d,n,\epsilon) \le O\left(d^{13/2}\,\epsilon\right)}$$

The proof has two parts.

First, we define a dynamical system based on operator scaling,

and show that $f(d,n,\epsilon) \le O(d^2\, n\, \epsilon)$.

Then, we do a smoothed analysis to remove the dependency on n.

***[Hamilton, Moitra 18]** $f(d,n,\epsilon) \le O(d^2\epsilon)$

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Alternating Algorithm

How to move an approximate frame to satisfy the two conditions exactly?

The problems is difficult with two conditions. It is easy with one condition.

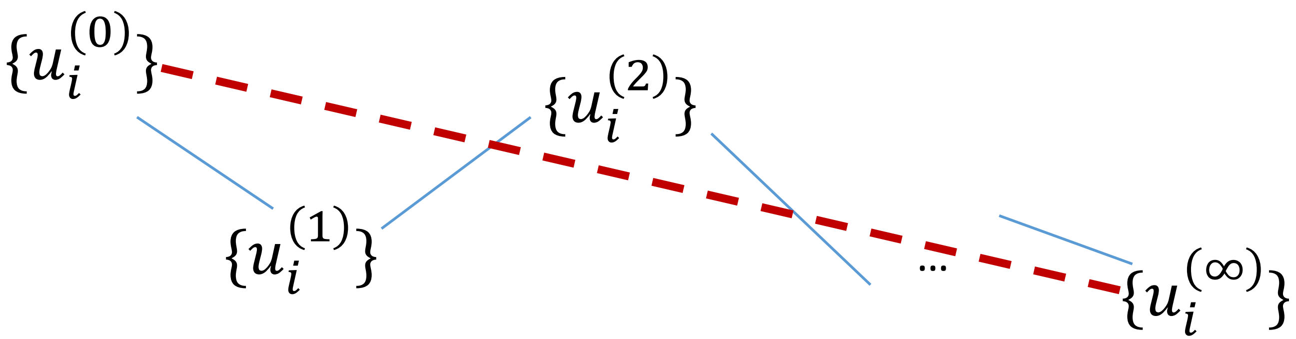
- To satisfy the equal norm condition, we just rescale the vectors.
- To satisfy the Parseval condition, we can set

$$u_i \leftarrow \left(\sum_{i=1}^{n} u_i u_i^T\right)^{-\frac{1}{2}} u_i \qquad \text{so that} \qquad \sum_{i=1}^{n} u_i u_i^T = I_d.$$

A natural algorithm is to alternate between these two steps and hope that it will converge to a solution satisfying both conditions.

First Idea

Our starting point is to bound the distance by the total movement in the alternating algorithm (assuming it converges):



This is a special case of the alternating algorithm for operator scaling, which was analyzed in [Gurvits 04, Garg-Gurvits-Oliveira-Wigderson 16].

Operator Scaling

An operator is a collection of matrices $U_1, \dots, U_k \in \mathbb{R}^{m \times n}$.

[Gurvits 04]

Given $U_1, \dots, U_k \in \mathbb{R}^{m \times n}$, we would like to find $L \in \mathbb{R}^{m \times m}$ and $R \in \mathbb{R}^{n \times n}$ such that if we define $V_i = LU_iR$ for $1 \le k \le n$ then

$$\sum_{i=1}^{k} V_i V_i^T = cnI_m \qquad \text{and} \qquad \sum_{i=1}^{k} V_i^T V_i = cmI_n$$

for some constant c.

We say an operator satisfying the two conditions <u>doubly balanced</u>.

Alternating Algorithm

Repeat the following two steps [Gurvits 04]:

- To satisfy the condition $\sum_{i=1}^{k} U_i U_i^T = I_m$, we set

$$U_i \leftarrow \left(\sum_{j=1}^{k} U_j U_j^T\right)^{-\frac{1}{2}} U_i$$

- To satisfy the condition $\sum_{i=1}^{k} U_i^T U_i = I_n$, we set

$$U_i \leftarrow U_i \left(\sum_{j=1}^{k} U_j^T U_j\right)^{-\frac{1}{2}}$$

A natural algorithm is to alternate between these two steps and hope that it will converge to a solution satisfying both conditions.

A simple reduction from frame scaling to operator scaling:

$$u_i \in \mathbb{R}^d \quad \rightarrow \quad U_i \equiv \begin{pmatrix} | & | & | \\ 0 & u_i & 0 \\ | & | & | \end{pmatrix} \in \mathbb{R}^{d\times n}$$

- The condition $\sum_{i=1}^{n} U_i U_i^T = I_d$ is the Parseval condition $\sum_{i=1}^{n} u_i u_i^T = I_d$.
- The condition $\sum_{i=1}^{n} U_i^T U_i = \frac{d}{n} I_n$ is the equal norm condition

$$\begin{pmatrix} \|u_1\|_2^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \|u_n\|_2^2 \end{pmatrix} = \frac{d}{n} I_n.$$

So we focus on this more general setting in this part of the talk.

The Operator Paulsen Problem

What is the best function $g(m, n, k, \epsilon)$ s.t. for any $U_1, \dots, U_k \in \mathbb{R}^{m \times n}$ with

$$(1-\epsilon)I_m \preccurlyeq \sum_{i=1}^{k} U_i U_i^T \preccurlyeq (1+\epsilon)I_m,\ (1-\epsilon)\frac{m}{n} I_n \preccurlyeq \sum_{i=1}^{k} U_i^T U_i \preccurlyeq (1+\epsilon)\frac{m}{n} I_n$$

there exist $V_1, \dots, V_k \in \mathbb{R}^{m \times n}$ with

$$\sum_{i=1}^{k} V_i V_i^T = I_m \quad \text{and} \quad \sum_{i=1}^{k} V_i^T V_i = \frac{m}{n} I_n$$

such that

$$\sum_{i=1}^{k} \|U_i - V_i\|_F^2 \le g(m, n, k, \epsilon) \boxed{\le m^2 n \epsilon}$$

Applications

Matrix Scaling:

- Preconditioning for linear solvers [Osborne 60]
- Optimal transportation [Wilson 69]
- Bipartite matching
- Deterministic approximation of permanents [Linial-Samorodnitsky-Wigderson 00]

Frame Scaling:

- Sign rank lower bound [Forster 02]
- Robust subspace recovery [Hardt-Moitra 13]
- Paulsen problem

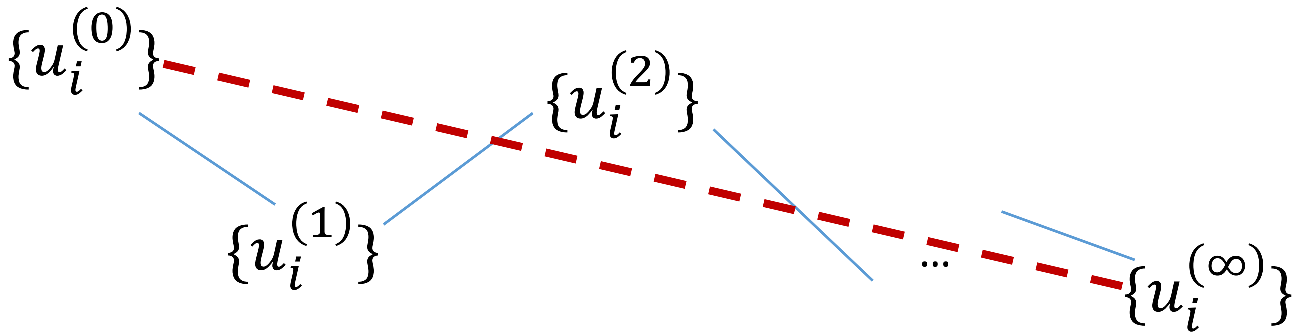
PSD scaling:

- Approximation of mixed discriminants [Gurvits-Samorodnitsky 02]

Operator Scaling:

- Computing non-commutative rank [Garg-Gurvits-Oliveira-Wigderson 16]
- Computing Brascamp-Lieb constants [Garg-Gurvits-Oliveira-Wigderson 17]
- Orbit intersection problem [AllenZhu-Garg-Li-Oliveira-Wigderson 18]

Issues in First Idea



There are examples which do not converge:

$$\binom{1}{0}, \binom{1}{0}, \binom{0}{1} \Leftrightarrow \binom{\sqrt{2}/2}{0}, \binom{\sqrt{2}/2}{0}, \binom{0}{1}$$

Even if it converges, the path could zig-zag a lot and

the total movement is much larger than the distance.

Error Measure

[Gurvits 04]

$$\boxed{\Delta = \frac{1}{m}\left\| sI_m - m\sum_{j=1}^{k} U_j U_j^T \right\|_F^2 + \frac{1}{n}\left\| sI_n - n\sum_{j=1}^{k} U_j^T U_j \right\|_F^2}$$

where $s = \sum_{i=1}^{k} ||U_i||_F^2$ is the **size** of the operator.

- Δ is zero if and only if the operator is doubly balanced.
- Can show that $\Delta \leq m^2\epsilon^2$.
- Focus on proving the total movement is $\leq mn\sqrt{\Delta} \leq m^2 n\epsilon$.

The dynamical system is moving in the direction that minimizes Δ.

Continuous Operator Scaling

Dynamical System: Do both steps simultaneously and continuously.

$$\frac{d}{dt}U_i = \left(sI_m - m\sum_{j=1}^{k} U_j U_j^T\right) U_i \; + \; U_i \left(sI_n - n\sum_{j=1}^{k} U_j^T U_j\right)$$

where $s = \sum_{i=1}^{k} \|U_i\|_F^2$ is the <u>**size**</u> of the operator.

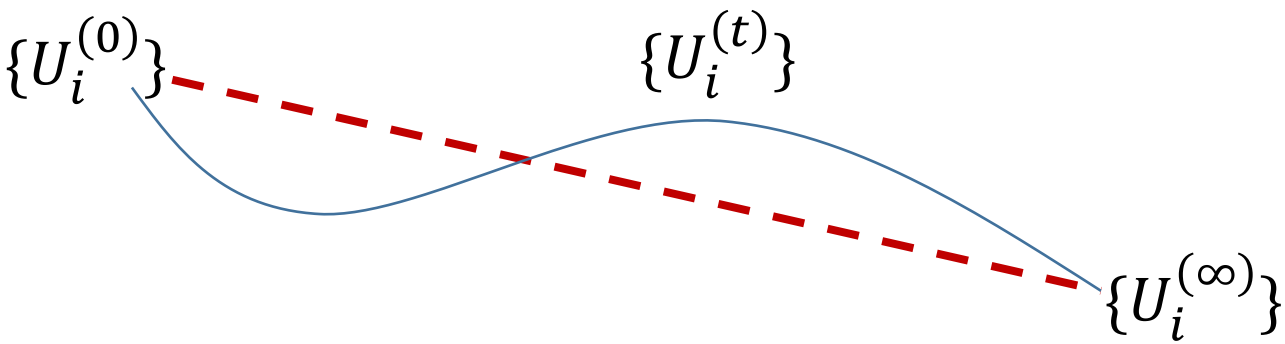
We find some nice identities to analyze the convergence.

Lemma 1. $\frac{d}{dt} s^{(t)} = -\Delta^{(t)}$.

Lemma 2. $\frac{d}{dt} \Delta^{(t)} = -\sum_{i=1}^{k} \left\| \frac{d}{dt} U_i^{(t)} \right\|_F^2$.

Claim. The dynamical system converges to a doubly balanced operator.

Total Movement



We again bound the final distance by the path length.

$$\left(\sum_{i=1}^{k} \left\| U_i^{(\infty)} - U_i^{(0)} \right\|_F^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{k} \left\| \int_0^\infty \frac{d}{dt} U_i^{(t)}\, dt \right\|_F^2\right)^{\frac{1}{2}}$$

distance

$$\leq \int_0^\infty \left(\sum_{i=1}^{k} \left\| \frac{d}{dt} U_i^{(t)} \right\|_F^2\right)^{\frac{1}{2}} dt \quad = \int_0^\infty \sqrt{-\frac{d}{dt}\Delta^{(t)}}\, dt$$

local movement

(triangle inequality) (Lemma 2)

Let T be the first time that $\Delta^{(T)} = \Delta^{(0)}/2$.

$$\left(\int_0^T \sqrt{-\frac{d}{dt}\Delta^{(t)}}\, dt\right)^2 \le \left(\int_0^T 1\, dt\right)\left(\int_0^T -\frac{d}{dt}\Delta^{(t)}\, dt\right) \le T\,\Delta^{(0)}.$$

We can complete the movement bound by a geometric sum argument.
So it remains to bound the **half time.**

Note Lemma 1 implies for all time up to T:

$$\boxed{\frac{d}{dt}\, s^{(t)} = -\Delta^{(t)} \le -\Delta^{(0)}/2}$$

Capacity

[Gurvits 04] Potential function to analyze operator scaling

$$\mathrm{cap}(\{U_i\}) = \inf_{X \in \mathbb{R}^{n\times n}, X \succ 0} \frac{m \det\left(\sum_{i=1}^{k} U_i X U_i^T\right)^{\frac{1}{m}}}{\det(X)^{\frac{1}{n}}}$$

Lemma 3. Capacity is unchanged over time.

Lemma 4. $s^{(t)} \geq \mathrm{cap}^{(t)} \geq s^{(t)} - mn\sqrt{\Delta^{(t)}}$.

We adapt the proof of Lemma 4 from [GGOW 16].

One implication is that $s^{(\infty)} = \mathrm{cap}^{(\infty)} = \mathrm{cap}^{(0)}$.

Bounding Half Time

<u>Half time</u>. Want to upper bound the first time T so that $\Delta^{(T)} = \Delta^{(0)}/2$.

Lemma 3. Capacity is unchanged over time.

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Lemma 4. $s^{(t)} \geq \mathrm{cap}^{(t)} \geq s^{(t)} - mn\sqrt{\Delta^{(t)}}$.

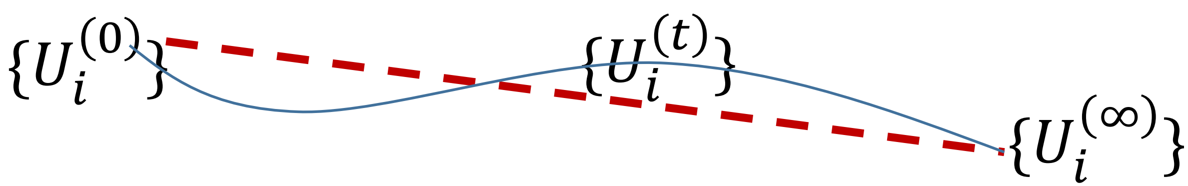
⇒ $s^{(T)} \geq \mathrm{cap}^{(T)} = \mathrm{cap}^{(0)} \geq s^{(0)} - mn\sqrt{\Delta^{(0)}}$

⇒ size of the operator decreases by at most $mn\sqrt{\Delta^{(0)}}$

Lemma 1. $\frac{d}{dt}\, s^{(t)} = -\Delta^{(t)}$. ⇒ size decreases by at least $\frac{1}{2}\,\Delta^{(0)} T$

⇒ $\boxed{T \leq \frac{2mn}{\sqrt{\Delta}}}$ ⇒ $\boxed{\text{total movement} \leq T\Delta \leq mn\sqrt{\Delta}.}$

Summary of Analysis



local movement

squared distance

$$\sum_{i=1}^{k} \left\| U_i^{(\infty)} - U_i^{(0)} \right\|_F^2 \le \left(\int_0^\infty \left(\sum_{i=1}^{k} \left\| \frac{d}{dt} U_i^{(t)} \right\|_F^2 \right)^{\frac{1}{2}} dt \right)^2$$

$$\le \left(\int_0^\infty \sqrt{-\frac{d}{dt} \Delta^{(t)}}\, dt \right)^2$$

Lemma 2

half time, geometric sum, Cauchy-Schwarz

$$\le T \Delta^{(0)}$$

$$\le mn\sqrt{\Delta^{(0)}}$$

Capacity argument, Lemma 1

L_2 vs L_∞

$$\le m^2 n \epsilon.$$

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Capacity and Total Movement

Part II can be understood as a reduction from total movement to capacity lower bound:

$$\mathrm{cap} \geq s - f(d,n,\Delta) \quad \underset{\text{part II}}{\Longrightarrow} \quad \mathrm{dist}^2 \leq f(d,n,\Delta).$$

In Part II, we proved $f(d,n,\Delta) \leq dn\sqrt{\Delta}$.

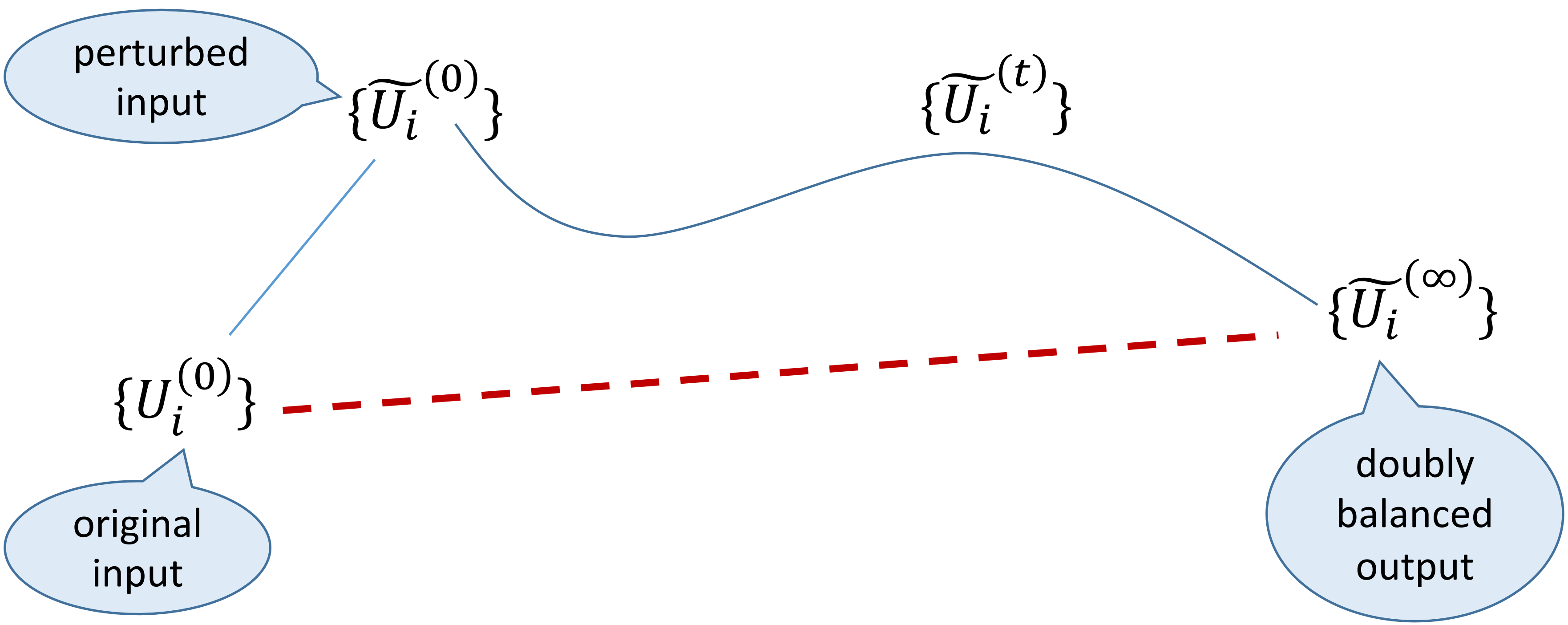
In Part III, we prove that $f(d,n,\Delta) \leq d^c\sqrt{\Delta}$ in “perturbed” instances.

Remark: Smoothed analysis only works in the frame setting, not (yet) in the operator setting.

Smoothed Analysis

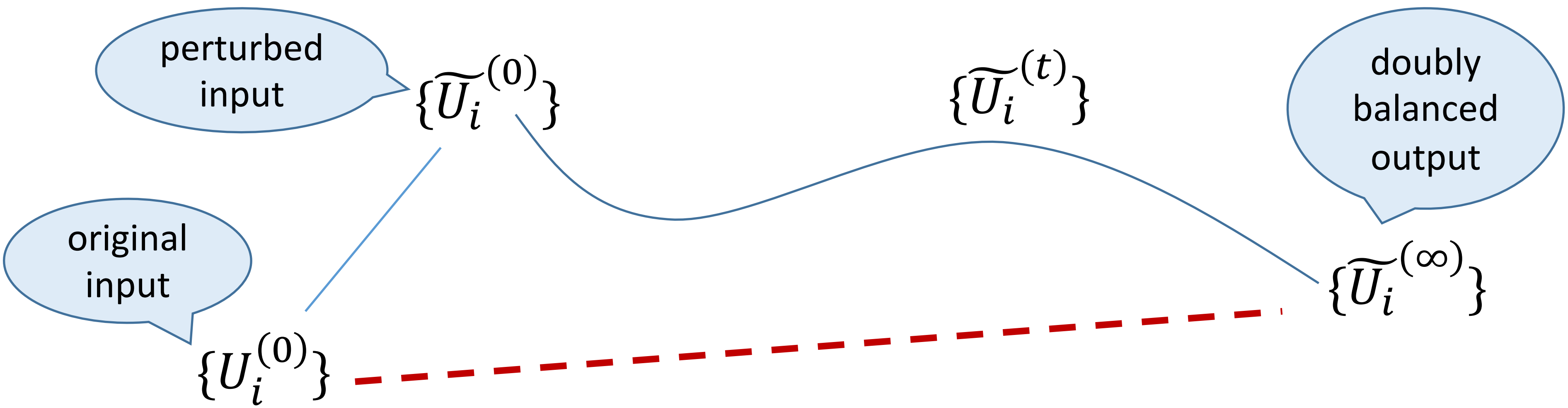
Intuition: operators with small capacity are rare.

Idea: perturb an operator, and apply the dynamical system.



Plan

movement in dynamical system $\leq f\left(d, n, \Delta(\widetilde{U})\right)$



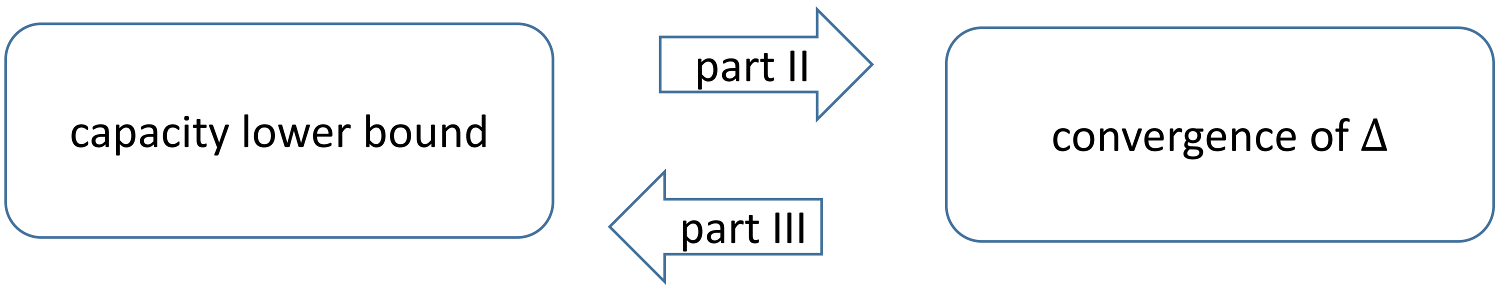
1. Upper bound the perturbation movement, i.e. $dist^2\left(\left\{U_i^{(0)}\right\}, \left\{\widetilde{U_i}^{(0)}\right\}\right)$.

2. Error won't increase too much, i.e. $\Delta(\widetilde{U}) \approx \Delta(U)$.

3. Improved capacity in perturbed instances, i.e. $f(d, n, \Delta) \leq d^c\sqrt{\Delta}$.

New Method in Capacity Lower Bound

New method: We use our dynamical system to bound matrix capacity.



$$\boxed{-\frac{d}{dt}\Delta \geq \mu\Delta \quad \Rightarrow \quad \mathrm{cap} \geq s - \frac{\Delta}{\mu}.}$$

So we need to show the fast convergence for the perturbed instances.

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Open Problems

New tools in bounding the mathematical quantities in scaling problems.

1. Bounding the condition number of scaling solutions. *
   - Used in fast algorithms for scaling problems.

2. Bounding (non-uniform) operator capacity
   - Equivalent in bounding Brascamp-Lieb constants.

3. Smoothed analysis of operator scaling

4. Generalization to Tensor scaling etc.