Geodesic Optimization and the Paulsen Problem in Frame Theory

Daniel Dadush, Akshay Ramachandran, Michael Walter

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We consider the following natural linear algebraic question:

Question 0.1. *Let $U = \{u_1, ..., u_n\} \subseteq \mathbb{R}^d$ be a spanning set of vectors satisfying*

$$\forall x \in \mathbb{R}^d : \frac{1-\varepsilon}{d} \leq \sum_{j=1}^{n} |\langle u_j, x\rangle|^2 \leq \frac{1+\varepsilon}{d}, \qquad \forall j \in [n] : \frac{1-\varepsilon}{n} \leq \|u_j\|_2^2 \leq \frac{1+\varepsilon}{n}. \tag{1}$$

What is the minimum dist$(V, U) := \sum_{j=1}^{n} \|v_j - u_j\|_2^2$ over all V exactly satisfying these conditions:

$$\forall x \in \mathbb{R}^d : \sum_{j=1}^{n} |\langle v_j, x\rangle|^2 = \frac{1}{d}, \qquad \forall j \in [n] : \|v_j\|_2^2 = \frac{1}{n}.$$

This is known as the Paulsen problem in frame theory and was listed as a major open problem ([4], [2]), for which little was known despite considerable effort. Frames satisfying the $\varepsilon = 0$ conditions above are known as doubly balanced frames. These give information theoretically optimal constructions for certain recovery tasks in signal processing and coding theory. On the other hand, doubly balanced frames can be difficult to construct explicitly, whereas it is easy to generate random frames satisfy 1 for some small ε with high probability. The Paulsen problem attempts to validate this approach by asking whether the frames satisfying 1 truly do approximate doubly balanced frames, and whether they can be rounded to nearby doubly balanced frames.

A priori, the distance bound may depend on the parameters (d, n, ε). There are known lower bounds showing that in the worst case $\text{dist}(V, U) \gtrsim \varepsilon$, whereas in practice the answer seems to be $\lesssim \varepsilon^2$. In a series of works [6, 5, 7], it was shown that the worst case lower bound ε is tight up to a constant factor.

Prior to these works, there were two partial results on the distance function [3, 1], which showed the distance bound $\text{poly}(d, n)\varepsilon^2$, but only in certain special cases. Note that this bound gives a better exponent for ε than the known worst case examples, and so cannot hold in general.

Project Goal: The procedures and analysis given in [3, 1] are slightly ad-hoc and unrelated to each other. We believe that the framework of group scaling and geodesic optimization (see [6, 5, 7]) can be used to simultaneously generalize and improve these results. This would give a unified and principled approach to the Paulsen problem. Our main goal is to exactly characterize the situations in which it is possible to improve the dependence on ε and prove a beyond worst case bound of the form ε^2. In particular, this would give theoretical justification for the improved performance of frames seen in practice.

References

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