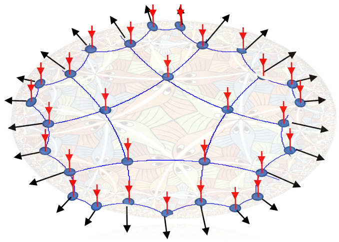
Quantum Information

Michael Walter



DRSTP PhD School, Trends in Theory, May 2024

Seek to leverage laws of QM for information processing...

communication cryptography

networks algorithms

quantum bits computation complexity

Quantum Information

entropy entanglement error correction

tensor networks quantum simulation

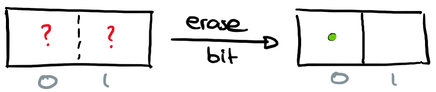
What do you find interesting?

...but also toolbox and language for studying quantum physics.

Physics vs Information: Thermodynamics

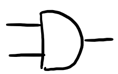
Irreversibility (2^{nd} law) vs coarse graining — Boltzmann, Gibbs, ...

Thermodynamics of computation: Cost of erasing a bit?



$W \geq kT \ln(2)$ — Landauer

Most logic gates are irreversible. Is there a fundamental cost to computing? No!



Bennett (1973): Efficient reversible computing is possible!

Physics vs Information: Computation

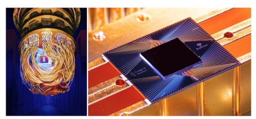
Simulating quantum physics difficult for classical computers.

Hilbert space is exponentially large

Why don't we build a quantum computer? — Feynman, Deutsch, ...

Shor's algorithm (1984): quantum computers may offer vast speedups for classical problems

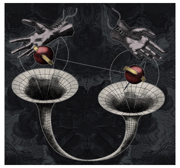
$N = pq$ in time poly($\log N$)



Google "quantum supremacy" experiment (2019)

Today, quantum simulation still one of most promising applications.

Physics vs Information: Language and Toolbox

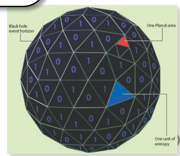


Quantum information is different: No cloning, uncertainty principle, Bell violations, entanglement, decoherence, …

QI offers language and toolbox to study and exploit these phenomena. Examples:

- Uncertainty principle ➔ quantum cryptography
- Bell violations ➔ device-independent control
- Entanglement ➔ many-body physics

In recent years, exciting research at interface of quantum information and many—body physics, from condensed-matter theory to QFT and gravity…



Plan

Goal: Discuss language, toolbox, key concepts of quantum information. Survey applications in many-body physics.

Page curve   tensor networks   Hayden-Preskill
nonlocality   replica trick   decoupling

We will start with slides and then switch to the black board. Throughout I will mention exercises that we can discuss during the tutorials. If you don't feel like taking notes: qi.rub.de/drstp2024

Please interrupt! ☺

If too slow (or too fast), please let me know.

If not detailed enough, please ask.

1. States, Channels, Entropy

Lecture Notes "Quantum Information Theory"
https://qi.rub.de/qi-book-draft.pdf

Quantum states

Density operators on some d-dimensional Hilbert space:

$$\rho = \sum_x p_x |\Psi_x\rangle\langle\Psi_x|$$

eigenvalues (p_x), eigenvectors ($|\Psi_x\rangle$)

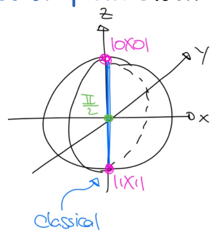
Pure states: $\rho = |\Psi\rangle\langle\Psi|$

Mixed states: model ensembles

$$\rho = \sum_i p_i \rho_i$$

(e.g. thermal states) and subsystems

States of qubit: Bloch ball



Entropy

$\rho = \Sigma_x\, p_x\, |\Psi_x\rangle\langle\Psi_x|$

Von Neumann entropy: $\boxed{S(\rho) = -\mathrm{tr}\, \rho \log_2 \rho}$ $= -\Sigma_x\, p_x \log p_x$

only depends on nonzero eigenvalues: $S(\rho) = S(U\rho U^\dagger)$

$0 \leq S(\rho) \leq \log(d)$

pure

$\rho = I/d$ "maximally mixed state"

"First law of entanglement"

$S(\rho + \delta\rho) = S(\rho) + \mathrm{tr}[\delta\rho\, K_\rho] + \dots$

Exercise

Modular Hamiltonian of ρ:

$K_\rho = -\log \rho$

state-dependent, often nonlocal

Renyi entropies and replica trick

Von Neumann entropy often difficult to compute ➔ Renyi entropies:

$$\boxed{S_n(\rho) = \frac{1}{1-n} \log \mathrm{tr}[\rho^n]} = \frac{1}{1-n} \log \Sigma_x\, p_x^{\,n}$$

$S_2(\rho) = -\log \mathrm{tr}[\rho^2]$
$S_1(\rho) = S(\rho)$
$S_0(\rho) = \log$ #nonzero eigenvalues

equal if ρ "flat" spectrum

$\log(d) \geq S_0(\rho) \geq S(\rho) \geq S_2(\rho) \geq \dots \geq 0$

Easy to calculate for integer n>1:

$\mathrm{tr}[\rho^2] = \mathrm{tr}[\rho^{\otimes 2}\, F]$ where $F\,|xy\rangle = |yx\rangle$ swap trick

$\mathrm{tr}[\rho^n] = \mathrm{tr}[\rho^{\otimes n}\, C_n]$ where $C_n\,|x_1x_2\dots\rangle = |x_2x_3\dots x_1\rangle$

Exercise

Joint systems

Reduced states of global states ρ_{AB} are given by partial trace:

$$\boxed{\rho_A = \mathrm{tr}_B(\rho_{AB})} \qquad \boxed{\langle a|\rho_A|a'\rangle = \sum_b \langle ab|\rho_{AB}|a'b\rangle}$$

➔ $\langle O_A\rangle_{\rho_A} = \langle O_A\rangle_{\rho_{AB}}$

Maximally entangled state (Bell/EPR pair): $|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$\Longrightarrow \rho_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$\Longrightarrow \rho_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{I}{2}$$ maximally mixed

Thus, pure states can have mixed reduced states. Conversely:

Fact: Any state ρ_A has a purification $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$.

Exercise

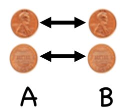
Correlations

We say that a state is correlated if not a product:

$\boxed{\rho_{AB} \neq \rho_A \otimes \rho_B}$

$\boxed{\langle O_A O'_B\rangle \neq \langle O_A\rangle\langle O'_B\rangle}$ for some pair of observables

Correlations can have quantum or classical origin:



Maximally entangled state: $|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Max. classically correlated: $\gamma_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$

In both cases, $\rho_A = \rho_B = I/2$, but $\rho_{AB} \neq I/2 \otimes I/2 = I/4$.

How to quantify correlations?

Mutual information

Mutual information: $I(A{:}B) = S(A) + S(B) - S(AB)$ ≥ 0, $= 0$ iff product

$I(A{:}B) = 2\log(d)$ iff maximally entangled $\quad |\Phi^+_{AB}\rangle = \frac{1}{\sqrt{d}}\sum_x |xx\rangle$

$I(A{:}B) = \log(d)$ if maximally classically correlated $\quad \gamma_{AB} = \frac{1}{d}\sum_x |xx\rangle\langle xx|$

Pinsker's inequality bounds correlation functions:

$$\left|\langle O_A O'_B\rangle - \langle O_A\rangle\langle O'_B\rangle\right| \leq \|O_A\|\|O'_B\|\sqrt{2\ln(2)\; I(A:B)}$$

Strong subadditivity (SSA): $I(A{:}BC) \geq I(A{:}B)$ never correlated more with subsystem

Fundamental, intuitive, not so easy to prove.

Quantum channels

What are the most general transformations of quantum states?

$\rho \longrightarrow \boxed{???} \longrightarrow \rho'$

Quantum channel: Any combination of unitary evolution, partial traces, adding auxiliary systems.

$\rho \rightarrow U\rho U^\dagger$

$\rho \rightarrow \rho\otimes\sigma$

$\rho_{A1A2} \rightarrow \rho_{A1}$

Mathematically: Completely positive trace-preserving maps.

Data processing inequality:

$I(A{:}B) \geq I(A'{:}B')$ ...if $\rho_{A'B'}$ obtained from ρ_{AB} by quantum channels A->A', B->B'.

Exercise: Prove this using SSA.

Application: Holevo bound

How many bits can we communicate by sending 1 qubit? infinitely many states!

Sender: $\{0,1\}^n \ni x \longrightarrow$ encoder $\xrightarrow{\rho(x)}$ (1 qubit state) Receiver: decoder $\longrightarrow y$

Challenge: Do not know optimal states nor optimal decoder! But note:

$$\rho_{XB} = 2^{-n}\sum_x |x\rangle\langle x|\otimes\rho(x) \quad \Longrightarrow \quad \rho_{XY} = 2^{-n}\sum_x |xx\rangle\langle xx|$$

...if can decode perfectly. Using the data processing inequality:

$$n = I(X:Y) \leq I(X:B) = S(B) - \sum_x p_x S(\rho(x)) \leq \log 2 = 1$$

Exercise: Verify this.

$n \leq 1$ ➔ no quantum advantage!

2. Entanglement

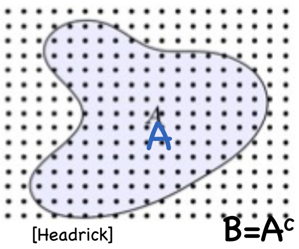
Literature: Lecture Notes "Quantum Information Theory"
https://qi.rub.de/qi-book-draft.pdf

Entanglement in pure states

We say that a pure state $|\Psi_{AB}\rangle$ is entangled if it is not a product:

$$\boxed{|\Psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\phi_B\rangle}$$

Why? If a pure state is classical, it must be a product: $|ab\rangle = |a\rangle \otimes |b\rangle$.
→ If it is not a product, it must be "quantumly correlated".



[Headrick]

Where does the entanglement come from? Typically due to to local interactions. → at low energies, entanglement between A and B concentrated near common boundary, "area law"

We will come back to this...

Schmidt decomposition

= SVD

$$\boxed{|\Psi_{AB}\rangle = \sum_{i=1}^{r} s_i \, |e_i\rangle \otimes |f_i\rangle}$$

Schmidt rank (r); Schmidt coefficients, >0 (s_i); orthogonal ($|e_i\rangle$); orthogonal ($|f_i\rangle$)

$$\boxed{\rho_A = \sum_{i=1}^{r} s_i^2 \, |e_i\rangle\langle e_i|} \qquad \boxed{\rho_B = \sum_{i=1}^{r} s_i^2 \, |f_i\rangle\langle f_i|}$$

➔ Reduced states have same eigenvalues, entropies, … and characterize entanglement:

$$\boxed{|\Psi_{AB}\rangle \text{ product} \Leftrightarrow r = 1 \Leftrightarrow \rho_A \text{ pure} \Leftrightarrow \rho_B \text{ pure}}$$

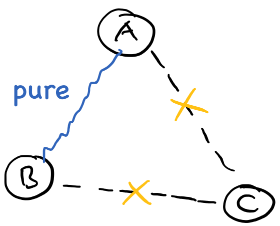
➔ Any two purifications of ρ_A are related by unitary on B

Extensions and Monogamy

Even if ρ_{AB} mixed: $\boxed{\rho_A \text{ pure} \rightarrow \rho_{AB} = \rho_A \otimes \rho_B}$

Take purification $|\Psi_{ABC}\rangle$ of ρ_{AB}. Since ρ_A pure, $|\Psi_{ABC}\rangle = |\Psi_A\rangle \otimes |\Psi_{BC}\rangle$.

This implies that pure state entanglement is monogamous:



AB pure ➔ AB uncorrelated with C

Monogamy: AB and AC cannot both be pure entangled.

In contrast, classical correlations can be arbitrarily shared.

Entanglement entropy

Schmidt decomposition suggests to quantify entanglement by the entropy of reduced states ➔ Entanglement entropy:

$$0 \leq \boxed{S_E = S(A) = S(B)} \leq \log d_A, \ \leq \log d_B$$

product state (0); maximally entangled ($\log d_A$, $\log d_B$)

<u>Interpretation:</u> Optimal conversion rate with Bell pairs:

$$\boxed{|\Psi_{AB}\rangle^{\otimes n} \xleftrightarrow{\text{LOCC}} \left(|00\rangle + |11\rangle\right)^{\otimes S_E n}}$$

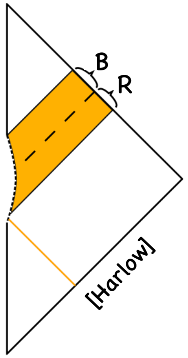
n→∞ copies
error → 0

$$\left.\begin{array}{l} \text{➔ entanglement transformations “reversible”} \\ \text{➔ Bell pair = unit of entanglement} \end{array}\right\} \text{for pure states}$$

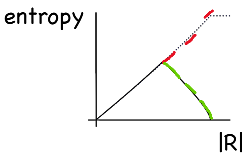
Application: Page curve

Suppose a black hole is created from infalling matter and we watch it evaporate.

R = Hawking radiation emitted up to some time
B = black hole = later Hawking radiation



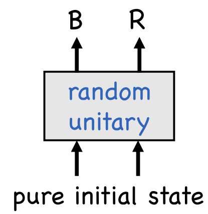
A semiclassical calculation suggests entropy of radiation increases until the end. But in a unitary theory, radiation will be pure once BH has evaporated...



Intuitively, early radiation is entangled with black hole, while late radiation is entangled with early radiation.

Application: Page curve

Model: black hole = random unitary
time = relative size of radiation R



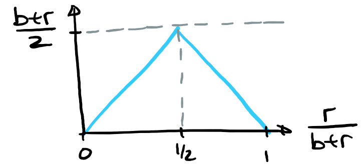
➔ $|\Psi_{BR}\rangle$ is a random pure state

B = b qubits
R = r qubits

Page theorem: For typical states,

$S_E = \min(b,r) - O(1)$

almost maximal!



Why random unitary? "Black holes are really complicated..."
Slightly more physical to consider a random Hamiltonian evolution.

Derivation of the Page formula

Idea: Lower-bound average Renyi-2 entropy $S_2(R)$ using swap trick.

Key formula: $\boxed{\overline{\Psi^{\otimes 2}} = \frac{I+F}{d(d+1)}}$ for random $\Psi=|\Psi\rangle\langle\Psi|$ in dimension d

Apply this to $|\Psi\rangle = |\Psi_{BR}\rangle$:

$$\overline{\Psi_{BR}^{\otimes 2}} = \frac{I_{BB} \otimes I_{RR} + F_{BB} \otimes F_{RR}}{d_B d_R (d_B d_R + 1)}$$

$$\Rightarrow \quad \underbrace{\overline{\operatorname{tr} \Psi_R^2} = \operatorname{tr} \overline{\Psi_{BR}^{\otimes 2}} F_{RR}}_{\text{swap trick}} \leq \frac{\operatorname{tr}\,(I_{BB} \otimes F_{RR} + F_{BB} \otimes I_{RR})}{d_B^2 d_R^2} = \frac{1}{d_R} + \frac{1}{d_B}$$

$$\Rightarrow \quad \overline{S_2(R)} \geq -\log \overline{\operatorname{tr} \Psi_R^2} \geq -\log\left(\frac{1}{d_R} + \frac{1}{d_B}\right) \geq \min(b,r) - 1$$

Jensen inequality

Exercise: Verify this.

Entanglement as a resource

What is entanglement good for? Four examples where entanglement enables otherwise impossible capabilities:

1) Superdense coding: communicate 2 bits by sending 1 qubit

   Holevo bound shows that impossible w/o entanglement

2) Teleportation: communicate 1 qubit by sending 2 bits

3) Violating Bell inequalities: produce non-classical correlations

4) Quantum cryptography: distill a shared secret key

It is also necessary for any quantum computational speedup.

Superdense coding

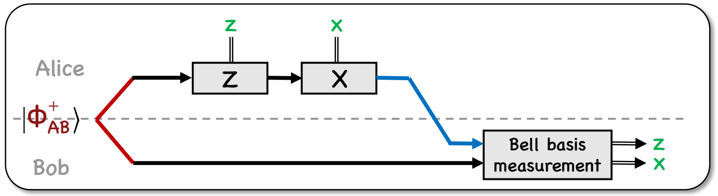
If Alice and Bob share EPR pair, they can use it to communicate 2 bits by sending 1 qubit!

"beating" the Holevo bound!

$$|\Phi^{(00)}_{AB}\rangle = (|00\rangle + |11\rangle)/\sqrt{2} = (I \otimes I)|\Phi^{+}_{AB}\rangle$$
$$|\Phi^{(01)}_{AB}\rangle = (|00\rangle - |11\rangle)/\sqrt{2} = (Z \otimes I)|\Phi^{+}_{AB}\rangle$$
$$|\Phi^{(10)}_{AB}\rangle = (|10\rangle + |01\rangle)/\sqrt{2} = (X \otimes I)|\Phi^{+}_{AB}\rangle$$
$$|\Phi^{(11)}_{AB}\rangle = (|10\rangle - |01\rangle)/\sqrt{2} = (XZ \otimes I)|\Phi^{+}_{AB}\rangle$$

"Bell basis"

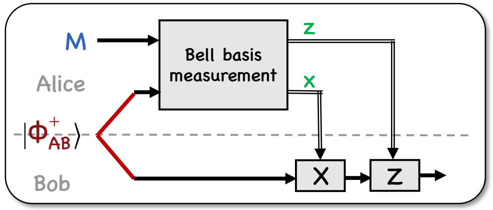
4 orthogonal states — created by local operation



Teleportation

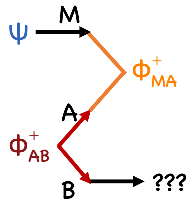
If Alice and Bob share EPR pair, they can use it to communicate 1 qubit by sending 2 bits!

#qubit states = ∞!



x, z completely random
➔ Alice learns nothing!

Why does it work? If outcome x=z=0, post-measurement state:

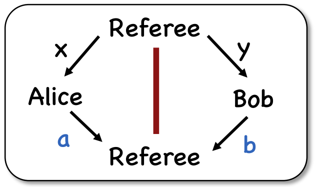


$$= \left(\langle\Phi^{+}_{MA}| \otimes I_B\right)\left(|\Psi_M\rangle \otimes |\Phi^{+}_{AB}\rangle\right)$$
$$= \frac{1}{2}\sum_{j,k}\left(\langle j|_M \otimes \langle j|_A \otimes I_B\right)\left(|\Psi_M\rangle \otimes |k\rangle_A \otimes |k\rangle_B\right)$$
$$= \frac{1}{2} I_{M\to B}|\Psi_M\rangle = \frac{1}{2}|\Psi_B\rangle$$

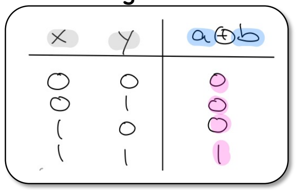
Bell inequalities as games

Clauser-Horne-Shimony-Holt

Alice and Bob play game:



winning condition:



Classical strategy: a=a(x), b=b(y)

$$a(0) \oplus b(0) \;\oplus\; a(0) \oplus b(1) \;\oplus\; a(1) \oplus b(0) \;\oplus\; a(1) \oplus b(1) \;\equiv\; 0$$

➔ will get at least one answer wrong: $\boxed{p_{win} \leq 3/4}$

This is a Bell inequality – a bound on classical correlations!

Nonlocality and quantum cryptography

If Alice and Bob share EPR pair, they can do better and achieve

$\boxed{p_{win,q} = 0.85..}$

Exercises

Tsirelson: optimal

crazy but true:
optimal strategy is "unique" (rigidity)

➔ can certify entanglement from correlations alone! ☺

Application: In quantum key distribution, Alice and Bob want to create a key secret from everyone else.

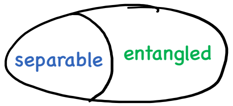
1) Alice and Bob play nonlocal game to ensure state is $|\Phi^{+}_{AB}\rangle$, by rigidity

2) Then $|\Psi_{ABE}\rangle = |\Phi^{+}_{AB}\rangle \otimes |\Psi_E\rangle$ by monogamy

3) Now measure to get random secret bit

Very rough sketch!

Bonus: Entanglement in mixed states



We say that a state is separable if it is a *mixture* of product states:

$$\rho_{AB} = \sum_i p_i \, \rho_A^{(i)} \otimes \rho_B^{(i)}$$

Motivation: classical correlations ≠ entanglement

Otherwise, the state is called entangled.

Separable states are precisely those that can be created by Local Operations and Classical Communication (LOCC).

Alice's laboratory		Bob's laboratory
arbitrary quantum operations	⇄ classical message	arbitrary quantum operations

Exercise: Show this.

That is, to create entanglement need to exchange quantum bits.

Entanglement in mixed states is hard

Recall that a state is separable if mixture of product states:

$$\rho_{AB} = \sum_i p_i \, \rho_A^{(i)} \otimes \rho_B^{(i)}$$

not canonical, typically non-orthogonal ↯

Bad news: NP-hard to check this condition

➔ no entanglement measure is faithful <u>and</u> easy to compute

A practical problem – it means meaningful calculations are difficult...

Similarly, multipartite entanglement. ρ_{AB} vs purification $|\Psi_{ABC}\rangle$

Bound entanglement

Can create any entangled state by LOCC given enough Bell pairs.

Bad news: Transformation usually irreversible.

$$|\Psi_{AB}\rangle^{\otimes n} \overset{\text{teleportation}}{\underset{\text{LOCC}}{\Longleftrightarrow}} \left(|00\rangle + |11\rangle\right)^{\otimes m}$$

conversion rates not equal ↯

There even exist "bound entangled" states such that no Bell pairs can be obtained from any number of copies!

➔ Zoo of entanglement measures: entanglement cost E_C, distillable entanglement E_D, ...

with different interpretations

Yet there are some practically useful criteria...

PPT criterion

Idea: Necessary for separability ⇔ sufficient for entanglement

Partial transpose (PT): $\langle ab|\rho_{AB}^{\Gamma}|a'b'\rangle = \langle ab'|\rho_{AB}|a'b\rangle$

"partial time reverse"

If ρ_{AB} separable then ρ_{AB}^{Γ} is again a density operator.

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \quad \Rightarrow \quad \rho_{AB}^{\Gamma} = \sum_i p_i \rho_A^{(i)} \otimes (\rho_B^{(i)})^T$$

PPT criterion: ρ_{AB}^{Γ} negative eigenvalues ➔ ρ_{AB} entangled

e.g. $|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\Gamma} \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$

Negativity

Partial transpose has tr=1. Thus, has negative eigenvalues ⇔ sum of absolute eigenvalues is > 1.

Negativity: $\boxed{N(\rho) = (\Sigma_i |\lambda_i| - 1)/2}$

Logarithmic negativity: $\boxed{E_N(\rho) = \log \Sigma_i |\lambda_i|}$

= 0 for separable states (but not only)

How to calculate?

1) Compute "Renyi negativities" $\mathrm{tr}\,(\rho_{AB}^{\Gamma})^{2n}$ and let $n \to 1/2$
2) Use replica trick: $\mathrm{tr}\,(\rho_{AB}^{\Gamma})^{2n} = \mathrm{tr}\,(\rho_{AB}^{\Gamma})^{\otimes 2n}\,(C_{2n} \otimes C_{2n}^{-1})$

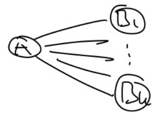
➔ Feasible in QFT and AdS/CFT!

Calabrese-Cardy-Tonni, Kusuki-Kudler-Flam-Ryu, …, Dong-Qi-W

Bonus: Entanglement vs monogamy

Say ρ_{AB} has k-extension if there is state σ on $AB_1 \ldots B_k$ with

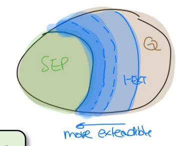
$$\boxed{\rho_{AB} = \sigma_{AB_1} = \cdots = \sigma_{AB_k}}$$



If ρ_{AB} separable then has k-extension for all k.

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \quad \Rightarrow \quad \sigma_{AB_1 \cdots B_k} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \otimes \ldots \otimes \rho_B^{(i)}$$

Conversely, if k-extension then $O(1/k)$ to separable.

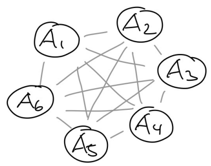


Criterion: $\boxed{\rho_{AB} \text{ separable} \Leftrightarrow \text{has k-extension for all k}}$

➔ Entanglement is monogamous also for mixed state!

Bonus: De Finetti theorem

Suppose that $A_1 \ldots A_n$ is permutation-symmetric. Then reduced states are close to mixtures of product states:



De Finetti Theorem: $\boxed{\rho_{A_1 \ldots A_k} \approx \int d\sigma\, p(\sigma)\, \sigma^{\otimes k}}$ if $k \ll n$

e.g. |00…0> + |11…1> and any k < n

➔ another version of monogamy

Physics application: Mean-field Hamiltonians have product ground states!

Bonus: Squashed entanglement

While mutual information is not a good entanglement measure, we can construct one using the conditional mutual information:

$$\boxed{I(A:B|C) = I(A:BC) - I(A:C)} = S(AC) + S(BC) - S(ABC) - S(C) \geq 0$$

Squashed entanglement: $\boxed{E_{sq}(A:B) = \frac{1}{2} \min_{\rho_{ABC}} I(A:B|C)}$

Intuition: entanglement = correlations that cannot be shared

Properties:

1) $0 \leq E_{sq} \leq \frac{1}{2} I(A:B) \leq \log \min(d_A, d_B)$
2) For pure states: $E_{sq} = \frac{1}{2} I(A:B) = S_E$
3) Separable ⇔ $E_{sq} = 0$
4) Monogamy: $E_{sq}(A:B) + E_{sq}(A:C) \leq E_{sq}(A:BC)$

Exercise: Show all but ← in 3.

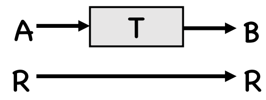
3. Quantum Channels and Dynamics

Recall: Quantum channels

Quantum channel: Any combination of unitary evolution, partial traces, adding auxiliary systems.

$\rho \to U\rho U^\dagger$
$\rho \to \rho \otimes \sigma$
$\rho_{AB} \to \rho_A$

Such maps even send states ρ_{AR} to states ρ_{BR} :



$\rho_{BR} = (T \otimes \mathrm{id})(\rho_{AR})$

completely positive & trace-preserving (CPTP)

Examples:

Basis measurement:

$M(\rho) = \Sigma_x \langle x|\rho|x\rangle \, |x\rangle\langle x|$

Depolarizing noise:

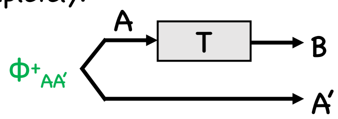
$D_f(\rho) = (1-f)\rho + f \, I/d$

Exercise: Check this.

Tools for quantum channels

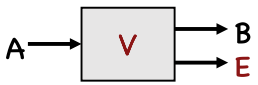
Choi state: characterizes channel completely!

$\Omega_{A'B} = (\mathrm{id} \otimes T)(\Phi^+_{A'A})$



Stinespring extension: Isometry V such that:

$T(\rho) = \mathrm{tr}_E(V\rho V^\dagger)$



➔ complementary channel: $T^c(\rho) = \mathrm{tr}_B(V\rho V^\dagger)$

Information lost to environment!

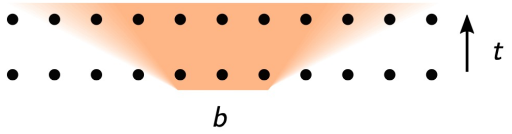
Together: Solve dynamics problems by (pure) state reasoning! ☺

Exercise: What are these for the measurement channel?

Local dynamics vs local Hamiltonians

Relativistic systems have sharp light cones (as do quantum circuits). Non-relativistic local Hamiltonian dynamics still have "approximate" light cones.

$$b(t) = e^{iHt} b e^{-iHt}$$

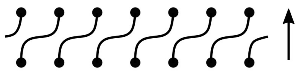


→ Lieb-Robinson velocity for short-range interactions

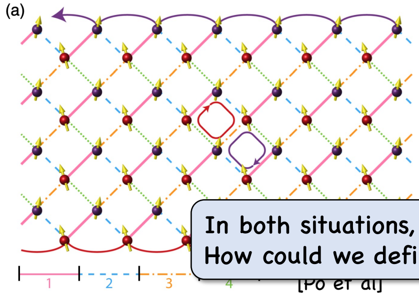
Question: Are local dynamics (evolutions with sharp or fuzzy light cones) always generated by local Hamiltonians?

Some interesting 1D dynamics

For example, can lattice translations be realized by a Hamiltonian?



This can even arise on the boundary of a 2D Hamiltonian dynamics:



[Po et al]

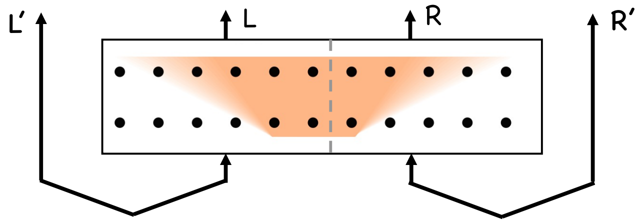
Floquet dynamics consisting of 4 layers of SWAP gates.

Trivial in the bulk, but has a "chiral edge".

In both situations, there is a clear **information flow**.
How could we define this in general?

Application: Information flow [Kitaev, GNVW, ..., RWW]

Consider dynamics of an infinite spin chain in 1D:



Step 1: Cut chain arbitrarily into halves.

Step 2: Consider Choi state $|\Omega_{L'R'LR}\rangle$

Step 3: Compute $\Delta = \frac{1}{2}\,(I(L':R) - I(R':L))$

net flow of quantum information, "index"

Amazingly, Δ is quantized and characterizes the dynamics!

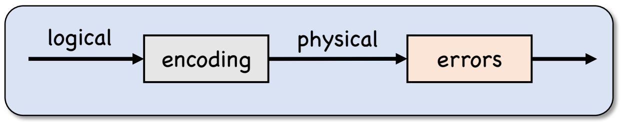
e.g. for qubit systems always an integer

Exercise?

e.g. dynamics is Hamiltonian iff $\Delta = 0$

Quantum error correction

When building quantum computers, we want to protect against errors.
To achieve this, redundantly encode "logical" into "physical" qubits:

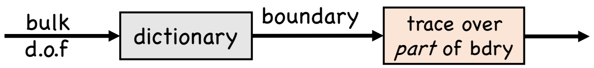


quantum channels!

Surprisingly, similar situations arise in fundamental physics:

Black holes

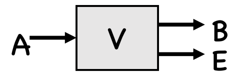
AdS/CFT:



Questions:

1) When can we in principle correct?
2) How to correct in practice?

Decoupling criterion



The question: Given a channel $T_{A\to B}$, when can we reverse it?

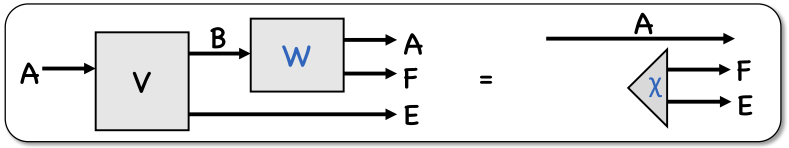
Decoupling criterion: Can reverse $T_{A\to B}$ if and only if the complementary channel $T^c_{A\to E}$ is constant.

i.e. $\Omega_{A'E} = \Omega_{A'} \otimes \chi_E$

$I(A':E) = 0$

➔ a very strong kind of "no cloning" statement

Idea: If reversible, there exists state $|\chi\rangle$ and isometry W such that:



or $|\Omega_{A'AEF}\rangle = |\Phi^+_{AA'}\rangle \otimes |\chi_{EF}\rangle$

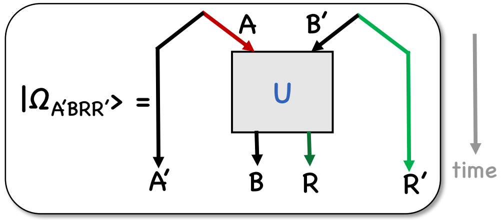
Exercise: Show this.

Application: Hayden-Preskill protocol

We again model an evaporating black hole by random unitary. **After Page time**, assume black hole maximally entangled with old radiation.

Now suppose Alice throws her diary into black hole.

How much further radiation do we need to collect so that we can recover diary?



That is, when can we decode A from RR′?

Need $\Omega_{A'B} \approx \Omega_{A'} \otimes \Omega_B$!

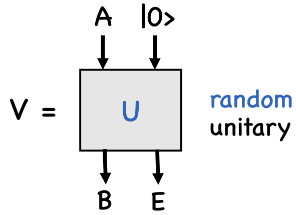
Answer: $d_R \gg d_A$

Little more than size of diary – independent of size of black hole! Black hole after Page time is like a mirror, information comes right out.

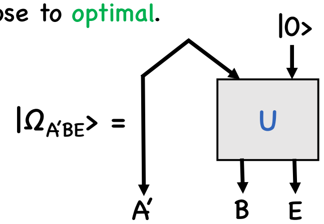
Exercise: Show this using the tool from the next slide, with B → R, E → B.

Tool: Decoupling Theorem

Similar questions had been studied in information theory, where it was observed that random codes are often close to optimal.



When can we decode A from B?



Need $\Omega_{A'E} \approx \Omega_{A'} \otimes \Omega_E$!

The following result addresses these kind of problems:

Decoupling Inequality: Let ρ_{ABE} state, U_{BE} random. Then:

$$\int dU_{BE} \, \| \mathrm{tr}_B(U_{BE}\, \rho_{ABE}\, U_{BE}^\dagger) - \rho_A \otimes I_E/d_E \|_1^2 \leq \frac{d_{AE}}{d_B}\, 2^{-S_2(ABE)}$$

4. Many-Body Entanglement and Tensor Networks (→ blackboard)

Literature: my TASI lecture notes

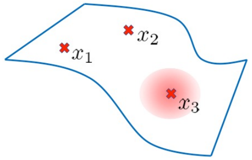
5. Quantum Information in QFT

Literature: Harlow (https://arxiv.org/abs/1409.1231), Headrick (https://arxiv.org/abs/1907.08126), Witten survey, my Kavli school slides

Quantum information & field theory

Do quantum information tools apply to quantum field theory?

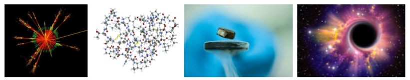
Challenge: Notions such as *subsystems*, *entropy*, *approximation*, *circuits* more subtle!



Why bother?

1. New insights: Bekenstein bound from relative entropy, renormalization as error correction, c-theorem from entropy...

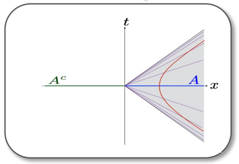
2. Quantum computers will be useful for simulating q. physics...

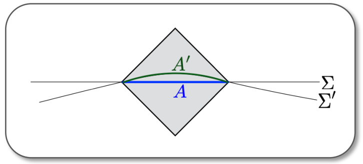


Feynman, Deutsch, ...

Can we simulate QFTs, or even theories of quantum gravity?

Subsystems in relativistic QFT





[Headrick]

Causal domain of A: D(A) = {p : every maximal causal curve through p intersects A}

Σ is Cauchy slice if acausal and D(Σ) = everything.

Time slice axiom:

Σ	⇔	global state	⇔	Hilbert space H
$A \subseteq \Sigma$	⇔	reduced state in D(A)	⇔	"$H = H_A \otimes H_B$"

D(A) = D(A′) ➔ ρ_A and $\rho_{A'}$ should be unitarily related

Correlations in QFT

Consider e.g. free scalar field with mass m in Minkowski space:

$$H = \int d^3x \; \pi(x)^2 + (\nabla\phi(x))^2 + m^2\, \phi(x)^2$$

$[\pi(x), \phi(y)] = i\delta^3(x-y)$

Correlation functions: $\langle\phi(x)\rangle = 0$

UV divergence

$$\langle\phi(x)\phi(y)\rangle \propto \begin{cases} |x-y|^{-2} & \text{if } |x-y| \ll \xi \\ \exp(-|x-y|/\xi) & \text{if } |x-y| \gg \xi \end{cases}$$

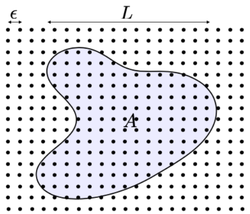
$\xi \sim 1/m$ correlation length

Amusing to compare with Bell pair:
$\langle X\rangle = \ldots = \langle Z\rangle = 0$
$\langle XX\rangle = \ldots = \langle ZZ\rangle = 1$

General form (short-distance divergence, long-distance decay) believed to hold in any relativistic QFT. If m=0, decay can be power law.

Entanglement in QFT

Correlation functions:
$$\langle\phi(x)\phi(y)\rangle \propto \begin{cases} |x-y|^{-2} & \text{if } |x-y| \ll \xi \\ \approx 0 & \text{if } |x-y| \gg \xi \end{cases}$$



[Headrick]

Thus, might expect that entanglement entropy satisfies an area law:

$$S(A) \propto |\partial A| \,/\, \varepsilon^{d-2}$$

UV cutoff

More generally, might expect that all divergences arise from local integrals over entangling surface ∂A.

All this assumes $\xi < \infty$. E.g. for CFTs in d=1+1, power law decay leads to $\log(|A|/\varepsilon)$ divergence, as we will discuss momentarily.

Entanglement in QFT

$H \neq H_A \otimes H_B$

Observables in A, B commute, but Hilbert space does not factorize.

cf. divergence across entangling surface

States not directly described by density operators
➔ Entanglement entropies not obviously well-defined

What can be said rigorously? → algebraic QFT literature, Witten's review

Reeh-Schlieder:

"$\{O_A |\Omega_{AB}\rangle\}$ dense"

Confusing? No, O_A will not be unitary!

Homework: Show that in finite dim any $|\Psi_{AB}\rangle$ can be written as $O_A |\Phi_{AB}^+\rangle$.

Relative entropies & various entanglement measures can be rigorously defined and computed/bounded e.g., still makes sense to distill EPR pairs!

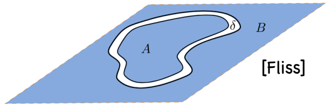
Bisognano-Wichmann: "modular Hamiltonian" of Rindler wedge

Entanglement Entropy in QFT?

We will proceed cavalierly since we must anyways regulate entanglement entropy to obtain finite answer.

General strategy: UV regulate and compute universal quantities

coefficient of $\log(|A|/\varepsilon)$



[Fliss]

relative entropy

$D(\rho\|\sigma) = \mathrm{tr}\, \rho\, (\log \rho - \log \sigma)$

mutual information I(A:B)

If A, B don't touch: "$H_{AB} = H_A \otimes H_B$"
➔ rigorously defined in QFT!

Intuition: divergences cancel

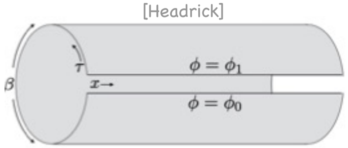
Euclidean path integrals

Let us consider states that are prepared by Euclidean path integrals.
E.g., unnormalized thermal state:

$\rho = e^{-\beta H}$

[Headrick]

$\langle \phi_0 | e^{-\beta H} | \phi_1 \rangle =$



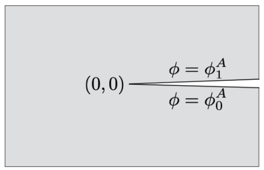
path integral on $[0,\beta] \times \Sigma$

For $\beta \to \infty$, obtain vacuum state.

➔ Reduced state of $A \subseteq \Sigma$:

$\rho_A = \mathrm{tr}_B\, e^{-\beta H}$

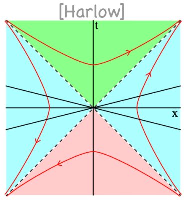
[Headrick]



path integral on plane with half-slit

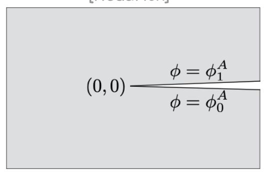
Rindler decomposition

[Harlow]



Minkowski space-time

[Headrick]



Euclidean path integral

Rindler wedges correspond to $A = [0,\infty)$ and $B = (-\infty,0]$.

Lorentz boost generator K acts by rotations in Euclidean signature ➔ $\rho_A = e^{-2\pi K}$ "thermal"

Similarly, Schmidt decomposition: $|\Omega_{AB}\rangle = \sum_i e^{-\pi \omega_i} |i'\rangle |i\rangle$

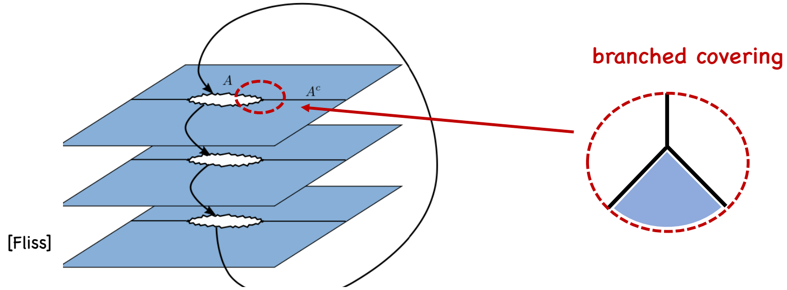
Amusing: If $|\Omega_{AB}\rangle$ were product ➔ "firewall" between A:B.

Entanglement entropy and replica trick

Using the replica trick, it is easy to compute Renyi entropies:

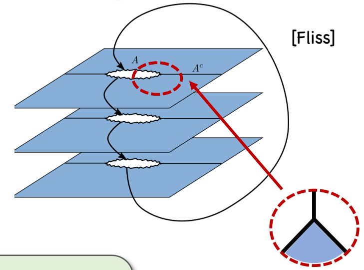
$$S_n(\rho) = \frac{1}{1-n} \log \frac{\mathrm{tr}[\rho^n]}{\mathrm{tr}[\rho]^n} = \frac{1}{1-n} (\log Z_n - n \log Z_1)$$

where $Z_n = \mathrm{tr}[\rho^n] = \mathrm{tr}[\rho^{\otimes n} C_n]$ is calculated by the following path integral:



Entanglement entropy for single interval

Can be explicitly computed for spherical regions in conformal field theory.



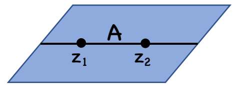
Cardy-Calabrese: In 1+1d CFT with central charge c,

$$S_n = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\frac{L}{\varepsilon} \qquad S = \frac{c}{3}\log\frac{L}{\varepsilon}$$

CFT fans: Prove this.

E.g. via 2-point function of twist operators in orbifold CFT:

$$Z_n = \langle \sigma_+(z_1)\sigma_-(z_2)\rangle_{\mathrm{CFT}^n/\mathbb{Z}_n}$$



Application: c-theorem

Casini-Huerta

Can use entanglement entropy to construct RG monotone and re-prove c-theorem.

$\boxed{c_{UV} \geq c_{IR}}$

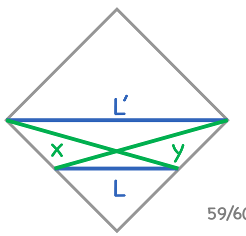
Suppose we deform "UV CFT" by relevant operator. Then:

$$\boxed{S(L \ll \xi) = \frac{c_{UV}}{3}\log\frac{L}{\varepsilon}} \qquad \boxed{S(L \gg \xi) = \frac{c_{IR}}{3}\log\frac{L}{\varepsilon'}}$$

Claim: $c(L) = 3\,L\,dS/dL$ interpolates c_{UV}, c_{IR} and decreases with L.

Key idea: Use strong subadditivity $S(AB) + S(BC) \geq S(ABC) + S(B)$.
Here:

$$\boxed{\underbrace{S(x) + S(y)}_{= 2S(\sqrt{LL'})} \geq S(L') + S(L)}$$



→ Exercise

Summary

Whirlwind tour through some key concepts and tools of quantum information, motivated by applications to theoretical physics.

- States, Channels, Entropy
- Entanglement of Pure and Mixed States
- Tools for Quantum Dynamics
- Quantum Information and QFT

Page curve

Hayden-Preskill

information flow

decoupling

replica trick

No time for quantum computing: circuits, algorithms, complexity, ... ☹

Slides: https://qi.rub.de/drstp2024