Geometric rank of tensors

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Different notions of rank

For a tensor $T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ there are many notions of rank

- tensor rank: $R(T) = \min\{r \mid T \leq I_r\}$
- flattening rank: e.g. $\mathrm{rk}(\mathbb{F}^{n_1 *} \to \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3})$
- border rank: $\underline{R}(T) = \min\{r \mid T \in \sigma_r\}$
- subrank: $Q(T) = \min\{r \mid I_r \leq T\}$
- ...

Today we will focus on another notion:

the geometric rank

Setting of today

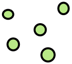
Work over algebraically closed $\mathbb{F}$, e.g. $\mathbb{F} = \mathbb{C}$.

Variety: the common zero set of a bunch of polynomial equations $\{x = (x_1, \ldots, x_n) \in \mathbb{F}^n \mid p_1(x) = \cdots = p_\ell(x) = 0\}$

We already saw some examples of varieties in the previous lectures...

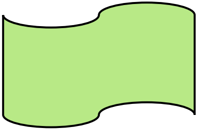
Dimension

Natural concept:



dim 0

dim 1



dim 2

Delicate concept: For an affine variety $X \subset \mathbb{F}^N$, the dimension of X is

$$\dim X = \text{the length of a maximal chain of irreducible subvarieties of } X.$$

The **codimension** of $X \subset \mathbb{F}^N$ is $\mathrm{codim} X = N - \dim X$.

Things to know about dimension

- for a linear space you already know how to compute dimensions from linear algebra
- if $X = \bigcup_i Y_i$ then $\dim X = \max \dim Y_i$
- if $Y \subseteq X$ then $\dim Y \leq \dim X$
- the dimension is additive for cartesian products
- A variety defined by as the common zero locus of just **one** equation $X = \{f = 0\} \subset \mathbb{F}^N$ is an hypersurface and $\dim X = N - 1$.

An example

Let $X = \{((x_1, x_2), (y_1, y_2)) \mid x_1 y_1 = 0, x_1 y_2 + y_2 x_1 = 0\} \subset \mathbb{F}^2 \times \mathbb{F}^2$.
We need to solve the system

$$\begin{cases} x_1 y_1 = 0, \\ x_1 y_2 + y_1 x_2 = 0 \end{cases} \iff \begin{cases} x_1 = 0 \text{ or } y_1 = 0 \\ x_1 y_2 + y_1 x_2 = 0 \end{cases}$$

- if $x_1 = 0$ then eq. 2 becomes $y_1 x_2 = 0$. This gives
    - $\{((x_1, x_2), (y_1, y_2)) \mid x_1 = 0, y_1 = 0\} = \mathbb{F}^1 \times \mathbb{F}^1$ or
    - $\{((0, 0), (y_1, y_2))\} = \{0\} \times \mathbb{F}^2$.

  In both cases we have 2 parameters of freedom, so the dimension of both components is 2
- if $y_1 = 0$ then the solutions are $\{y_1 = x_1 = 0\} = \mathbb{F}^1 \times \mathbb{F}^1$ and $\{y_1 = 0, y_2 = 0\} = \mathbb{F}^2 \times \{0\}$. Again dim 2.

Hence, $X = \{x_1 = x_2 = 0\} \cup \{y_1 = y_2 = 0\} \cup \{x_1 = y_1 = 0\}$,
$\dim X = 2$ and $\operatorname{codim} X = 4 - 2 = 2$.

The geometric rank of a tensor

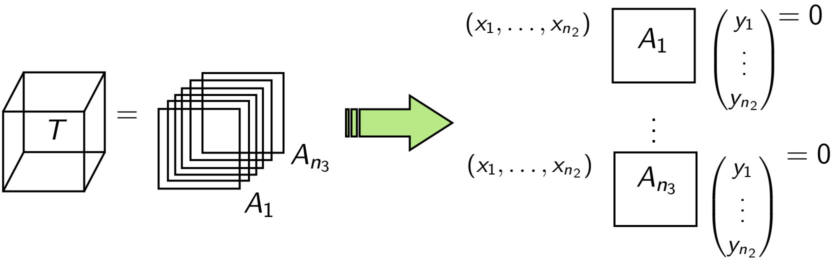
Kopparty-Moskowitz-Zuiddam 2022

Let $T = (t_{i,j,k}) \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$. Fix the 3^{rd} factor and take $A_1 = (t_{i,j,1}), \ldots, A_{n_3}(t_{i,j,n_3}) \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$.

The **geometric rank of T** is

$$\mathrm{GR}(T) := \mathrm{codim}\{(x, y) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \mid x^T A_1 y = \cdots = x^T A_{n_3} y = 0\}.$$

The codimension of the solutions of a system of quadratic equations:



Example

Consider the W-state

$$T = e_2 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_2 \in \mathbb{F}^{2\times 2\times 2}$$
$$= \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right).$$

To compute $\mathrm{GR}(T)$ we need to consider $x^T A_1 y = 0$ and $x^T A_2 y = 0$, i.e.

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$
$$x_1 y_2 + x_2 y_1 = 0 \text{ and } x_1 y_1 = 0.$$

In the previous example we computed $\mathrm{codim}\{(x, y) \mid x_1 y_2 + x_2 y_1 = x_1 y_1 = 0.\} = 2 = \mathrm{GR}(T)$.

Let us look at the definition again

Let $T = (t_{i,j,k}) \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$. Fix the $\mathbf{3}^{\text{rd}}$ **factor** and take $A_1 = (t_{i,j,1}), \ldots, A_{n_3}(t_{i,j,n_3}) \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$.

$\mathrm{GR}(T) := \mathrm{codim}\{(x, y) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \,|\, x^T A_1 y = \cdots = x^T A_{n_3} y = 0\}$.

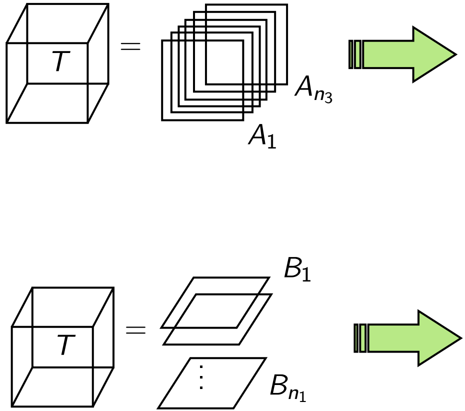
Fix the $\mathbf{1}^{\text{st}}$ **factor** and take slices $B_1 = (t_{1,j,k}), \ldots, B_{n_1} = (t_{n_1,j,k})$.

We can look at

$$\mathrm{codim}\{(x, y) \in \mathbb{F}^{n_2} \times \mathbb{F}^{n_3} | x^T B_1 y = \cdots = x^T B_{n_1} y = 0\}$$

Do they have the same codimension?

Is GR well defined?



$$(x_1, \ldots, x_{n_1}) \boxed{A_1} \begin{pmatrix} y_1 \\ \vdots \\ y_{n_2} \end{pmatrix} = 0$$

$$\vdots$$

$$(x_1, \ldots, x_{n_1}) \boxed{A_{n_3}} \begin{pmatrix} y_1 \\ \vdots \\ y_{n_2} \end{pmatrix} = 0$$

$$(x_1, \ldots, x_{n_2}) \boxed{B_1} \begin{pmatrix} y_1 \\ \vdots \\ y_{n_3} \end{pmatrix} = 0$$

$$\vdots$$

$$(x_1, \ldots, x_{n_2}) \boxed{B_{n_1}} \begin{pmatrix} y_1 \\ \vdots \\ y_{n_3} \end{pmatrix} = 0$$

To answer this question, it is convenient to look at $T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$ also as a multilinear map

$$T : \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \times \mathbb{F}^{n_3} \to \mathbb{F}$$
$$(x, y, z) \mapsto \sum_{i,j,k} t_{i,j,k} x_i y_j z_k.$$

In this way we can rephrase the geometric rank as

$$\begin{aligned} \mathrm{GR}(T) &= \mathrm{codim}\{(x, y) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \mid T(x, y, z) = 0 \forall z\} \\ &= \mathrm{codim}\{(x, y) \in \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \mid T(x, y, \cdot) = 0\}, \end{aligned}$$

where $T(x, y, \cdot)$ is the vector containing the slices.

Notice that

$$\{(x,y)\,|\,x^TA_iy=0 \text{ for all } i\} = \bigtimes_{x\in\mathbb{F}^{n_1}} \{y\in\mathbb{F}^{n_2}\,|\,x^TA_iy=0 \text{ for all } i\}.$$

Moreover, for fixed x we have

$$\dim\{y\in\mathbb{F}^{n_2}\,|\,x^TA_iy=0 \text{ for all i}\} = \dim\ker\begin{bmatrix} x^TA_1 \\ \vdots \\ x^TA_{n_3}\end{bmatrix} = \textit{corank} \text{ Big M}.$$

What is this big matrix? Call it $T(x,\cdot,\cdot)$.

Define $W_i=\{x\in\mathbb{F}^{n_1}\,|\,\textit{corank}\,T(x,\cdot,\cdot)=i\}$ and notice that the W_i are a partition of $\mathbb{F}^{n_1}$. So $\{(x,y)\,|\,x^TA_iy=0 \text{ for all } i\}$ equals

$$\bigcup_i\{(x,y)\in W_i\times\mathbb{F}^{n_2}\,|\,x^TA_1y=\cdots=x^TA_{n_3}y=0\}$$

Hence, $\dim\{(x,y)\,|\,x^TA_iy=0 \text{ for all } i\} = \max_i\{\dim W_i+i\}$.

$$W_i = \{x \in \mathbb{F}^{n_1} \mid corank T(x,\cdot,\cdot) = i\}$$

Now, since we are looking for codimension, we have

$$\begin{aligned}
\mathrm{GR}(T) &= codim\{(x,y) \mid T(x,y,\cdot) = 0\} \\
&= n_1 + n_2 - \max_i\{\dim W_i + i\} \\
&= min_i\{n_1 + n_2 - (\dim W_i + i)\} \\
&= min_i\{n_1 - \dim\{x \mid \mathrm{rk}\, T(x,\cdot,\cdot) = n_2 - i\} + n_2 - i\} \\
&= min_j\{codim\{x \mid \mathrm{rk}\, T(x,\cdot,\cdot) = j\} + j\}.
\end{aligned}$$

It only depends on x! So if we start with $codim\{(x,z) \mid T(x,\cdot,z) = 0\}$ we get the same!

$\implies$ GR well defined!

On the big matrix

We were looking at

$$\begin{bmatrix} x^T A_1 \\ \vdots \\ x^T A_{n_3} \end{bmatrix} = \begin{bmatrix} \sum_i t_{i,1,1} x_i & \dots & \sum_i t_{i,n_2,1} x_i \\ \vdots & & \vdots \\ \sum_i t_{i,1,n_3} x_i & \dots & \sum_i t_{i,n_2,n_3} x_i \end{bmatrix}$$

$$= \begin{bmatrix} x^T B_1 & \dots & x^T B_{n_2} \end{bmatrix},$$

where $A_r = (t_{i,j,r})$ and $B_s = (t_{i,s,j})$.

That is why we were simply calling it $T(x, \cdot, \cdot)$.

GR for many factors

The geometric rank can be defined for an arbitrary number of factors. For $T \in \mathbb{F}^{n_1 \times \cdots \times n_k}$, $\mathrm{GR}(T)$ is the codimension of

$$\{(x_1, \ldots, x_{k-1}) \in \mathbb{F}^{n_1} \times \cdots \times \mathbb{F}^{n_{k-1}} \mid T(x_1, \ldots, x_{k-1}, x_k) = 0 \forall x_k\}.$$

What happens in the case of matrices?

In all notions seen so far ($R(T), Q(T)\dots$) when restricting to the case of matrices, all these notions correspond to the well known rank of matrices rk.

Does this happens also for GR?

Take $T = (t_{i,j}) \in \mathbb{F}^m \times \mathbb{F}^n$. We have

$$\begin{aligned}\mathrm{GR}(T) &= \mathrm{codim}\{(x,y) \,|\, \forall x \;\; T(x,y) = 0\} \\ &= n - \dim\{y \in \mathbb{F}^n \,|\, \sum_j t_{1,j} y_j = \cdots = \sum_j t_{m,j} y_j = 0\} \\ &= n - \dim\{y \,|\, Ty = 0\} = n - \dim \ker T = \mathrm{rk}\, T.\end{aligned}$$

Properties of GR

- if $S \leq T$ then $\mathrm{GR}(S) \leq \mathrm{GR}(T)$
  - first prove that $(A, I, I) \cdot T$ has GR less or equal than $\mathrm{GR}(T)$, then chain with $(I, B, I) \cdot T$ and $(I, I, C) \cdot T$.
- $\mathrm{GR}(S \oplus T) = \mathrm{GR}(S) + \mathrm{GR}(T)$, for $S \in \mathbb{F}^{m_1 \times m_2 \times m_3}$, $T \in \mathbb{F}^{n_1 \times n_2 \times n_3}$
  - If S and T have slices $A_1, \ldots, A_{m_3}$ and $B_1, \ldots, B_{n_3}$ then $S \oplus T$ has slices $A_i \oplus 0$ and $0 \oplus B_j$ and the variables do not interact with each others.
- sub additive element wise
  - Since $S + T \leq S \oplus T$ and $\mathrm{GR}(S \oplus T) = \mathrm{GR}(S) + \mathrm{GR}(T)$.
- GR is not submultiplicative under kronecker product (e.g. M_{nnn})

Example M_{222}

We already saw that $M_{222} \sim M_{211} \boxtimes M_{121} \boxtimes M_{112}$.
It is easy to prove that $\mathrm{GR}(M_{112}) = \mathrm{GR}(M_{121}) = \mathrm{GR}(M_{211}) = 1$:

$$M_{112} = e_1 \otimes (e_1 \otimes e_1 + e_2 \otimes e_2) \text{ only one slice } \implies \mathrm{GR} = 1$$

Let us compute now GR of $M_{2,2,2}$

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

We need to find the dimension of

$$\{x_1 y_1 + x_2 y_3 = x_1 y_2 + x_2 y_4 = x_3 y_1 + x_4 y_3 = x_3 y_2 + x_4 y_4 = 0\}.$$

This is given by 3 pieces each having dimension 5
$\implies \mathrm{GR}(M_{2,2,2}) = 3$.

$\mathrm{GR}(I_r)$

Recall that $I_r = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i \in \mathbb{F}^{r \times r \times r}$ and let us compute $\mathrm{GR}(I_r)$.

For $r = 1$ we have to look at $\{xy = 0\} = \{x = 0\} \cup \{y = 0\}$. So $\dim\{(x, y) \in \mathbb{F} \times \mathbb{F} \mid xy = 0\} = 1$, therefore $\mathrm{GR}(I_1) = 2 - 1 = 1$.

In general we have $\mathrm{GR}(I_r) = r$.

Indeed, $I_r = \oplus^r I_1$ and we have additivity under direct sum, So $\mathrm{GR}(I_r) = r\mathrm{GR}(I_1) = r$.

You can also directly compute that $\{(x, y) \mid x_1 y_1 = \cdots = x_r y_r = 0\}$ has dimension r and so $\mathrm{GR}(I_r) = 2r - r = r$.

Comparing GR with other ranks

We want to prove that

$$Q(T) \leq \mathrm{GR}(T) \leq SR(T).$$

- Assume $Q(T) = s$, so $I_s \leq T$. We just computed that $\mathrm{GR}(I_s) = s$ and we know that GR is monotone under restriction: $\mathrm{GR}(I_s) \leq \mathrm{GR}(T)$. So

$$Q(T) = s = \mathrm{GR}(I_s) \leq \mathrm{GR}(T).$$

- First, notice that if $SR(T) = 1$ then $\mathrm{GR}(T) = 1$. Hence $\mathrm{codim}\{(x, y) \,|\, x^T Ax = 0\} = n_2 + n_3 - (n_2 + n_3 - 1) = 1 = \mathrm{GR}(T)$. Assume $SR(T) = r$, so $T = \sum_i^r T_i$ where each T_i has slice rank 1.So $\mathrm{GR}(T_i) = 1$ for all i. We conclude by element wise subadditivity.

Application to hypergraphs

Undirected uniform hypergraph $H := (V, E)$

$$V = \{1, \dots, n\} \text{ and } E \subset 2^V \text{ such that } \#e = 3 \quad \forall e \in E.$$

We associate to H a tensor $T = (t_{i,j,k}) \in \mathbb{F}^{n \times n \times n}$ as follows:

$$t_{i,j,k} := \begin{cases} 1 & \text{if } \{i,j,k\} \in E \text{ or } i = j = k \\ 0 & \text{otherwise.} \end{cases}$$

The **independence number** of H is $\alpha := \#$ largest set of vertices containing no edges of H.
The value α can be bounded by

- subrank (**hard** to compute)
- geometric rank (**easy** to compute).

Some more applications

For $T \in \mathbb{F}^{n\times n\times n}$, the **border subrank** is defined as

$$\underline{Q} = \max r \text{ such that } I_r \in \overline{GL_n \times GL_n \times GL_n \cdot T}$$

and

$$\mathrm{GR}(T) \geq \underline{Q}(T).$$

As a consequence, the authors prove that $\underline{Q}(M_{n,n,n}) = \lceil 3/4n^2 \rceil$.

Some references on the topic

S. Kopparty, G Moshkovitz, J Zuiddam: *Geometric rank of tensors and subrank of matrix multiplication*. Discrete Analysis, 2023.

A more geometric perspectivre

- R Geng and J M Landsberg. *On the geometry of geometric rank*. Algebra and Number Theory, 16(5):1141-1160, 2022.
- R Geng. *Geometric rank and linear determinantal varieties*. European Journal of Mathematics 9.2 (2023): 23.

Let us focus on symmetric tensors

An important class of tensors $\mathbb{F}^{n\times n\times n}$ is the one of *symmetric* tensors.
A tensor $T = (t_{i,j,k}) \in \mathbb{F}^{n\times n\times n}$ is symmetric if

$$t_{i,j,k} = t_{\sigma(i),\sigma(j),\sigma(k)}, \text{ for all } \sigma \in \mathfrak{S}_3.$$

Symmetric tensors actually form a vector space that is usually denoted as

$$\mathrm{Sym}^3\mathbb{F}^n = \{T \in \mathbb{F}^{n\times n\times n} \mid T \text{ is symmetric}\}.$$

- The W-state $T = e_2 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_2$ is symmetric.

Symmetric rank

All notions of tensors seen so far can be *adapted* for the particular case of symmetric tensors. For $T \in \mathrm{Sym}^3\mathbb{F}^n$ we can look at

$$R(\cdot) = \min\{r \mid T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i\} \qquad \textbf{tensor rank}$$

but also at

$$R_{sym}(\cdot) := \min\{r \mid T = \sum_{i=1}^{r} v_i \otimes v_i \otimes v_i\} \qquad \textbf{Waring rank}$$

We have

$$R \leq R_{sym}.$$

Understanding when equality holds is the well-known **Comon's problem**.

Symmetric geometric rank

Also for the geometric rank we can consider its symmetrization.
Recall that for $T \in \mathrm{Sym}^3\mathbb{F}^n$,

$$\mathrm{GR}(T) = \mathrm{codim}\{(x, y) \in \mathbb{F}^n \times \mathbb{F}^n \mid x^T A_1 y = \cdots = x^T A_n y = 0\}.$$

$$x^T A_i y \rightsquigarrow x^T A_i x$$

Denote by $A_1 \ldots, A_n$ the slices of $T \in \mathrm{Sym}^3(\mathbb{F}^n)$. The **symmetric geometric rank of** T is

$$\mathrm{SGR}(T) := \mathrm{codim}\{x \in \mathbb{F}^n | x^T A_1 x = \cdots = x^T A_n x = 0\}.$$

Symmetric geometric rank I

For $T \in \mathrm{Sym}^3(\mathbb{F}^n)$, A_i slice of T

$$\mathrm{SGR}(T) := \mathrm{codim}\{x \in \mathbb{F}^n | x^T A_1 x = \cdots = x^T A_n x = 0\},$$

not very revealing...

But hey, symmetric tensors are homogeneous polynomials!

$$\mathrm{Sym}^3\mathbb{F}^n \xrightarrow{\sim} \mathbb{C}[x_1, \dots, x_n]_{(3)}$$
$$T = (t_{i,j,k}) \mapsto \sum t_{i,j,k} x_i x_j x_k =: F.$$

Moreover,

$$x^T A_i x \cong \frac{\partial F}{\partial x_i}.$$

Example

for $x^T A_i x \cong \frac{\partial F}{\partial x_i}$

$T = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$, or equivalently
$F = x_1 x_1 x_2 + x_1 x_2 x_1 + x_2 x_1 x_1 = 3x_1^2 x_2$.

$$\begin{aligned} T &= e_1 \otimes (e_2 \otimes e_1 + e_1 \otimes e_2) & + \quad & e_2 \otimes e_1 \otimes e_1 \\ &= e_1 \otimes A_1 & + \quad & e_2 \otimes A_2 \\ &= e_1 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & + \quad & e_2 \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\frac{\partial F}{\partial x_1} = 6x_1 x_2 = 3 \cdot x^T A_1 x \qquad\qquad \frac{\partial F}{\partial x_2} = 3x_1^2 = 3 \cdot x^T A_2 x$$

$x^T A_i x$ is **equal** to $\frac{\partial F}{\partial x_i}$ up to a non zero scalar.

Symmetric geometric rank II

Let F be **the** homogeneous polynomial associated to T,

$$\begin{aligned}\mathrm{SGR}(T) &:= \operatorname{codim}\{x \in \mathbb{F}^n \mid x^T A_1 x = \cdots = x^T A_n x = 0\}\\ &= \operatorname{codim}\left\{x \in \mathbb{F}^n \;\middle|\; \frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0\right\}.\end{aligned}$$

Symmetric geometric rank II

Let F be **the** homogeneous polynomial associated to T,

$$\mathrm{SGR}(T) := \mathrm{codim}\{x \in \mathbb{F}^n \mid x^T A_1 x = \cdots = x^T A_n x = 0\}$$
$$= \mathrm{codim}\left\{x \in \mathbb{F}^n \;\middle|\; \frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0\right\}.$$

Recall:

- The zero locus $X_F = \{F = 0\} \subset \mathbb{F}^n$ of F is an hypersurface.
- A point $p \in \mathbb{F}^n$ is **singular** for X_F if $F(p) = 0$ and $\frac{\mathrm{d}F(p)}{\mathrm{d}x_i} = 0$ for all i.
- The singular locus of X_F is $\mathrm{Sing}(F) = \{\frac{\mathrm{d}F}{\mathrm{d}x_0} = \cdots = \frac{\mathrm{d}F}{\mathrm{d}x_n} = 0\}$.

Symmetric geometric rank II

Let F be **the** homogeneous polynomial associated to T,

$$\begin{aligned}\mathrm{SGR}(T) &:= \operatorname{codim}\{x \in \mathbb{F}^n \mid x^T A_1 x = \cdots = x^T A_n x = 0\} \\ &= \operatorname{codim}\left\{x \in \mathbb{F}^n \;\middle|\; \frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0\right\}.\end{aligned}$$

$$\mathrm{SGR}(T) := \operatorname{codim}_{\mathbb{F}^n}(\mathrm{Sing}(F)).$$

Already well defined!
Already generalizable to an arbitrary number of factors.

Relation between GR and SGR

For a $T \in \mathrm{Sym}\mathbb{F}^3 \subset \mathbb{F}^{n\times n\times n}$ we have

$$\mathrm{SGR}(T) \leq \mathrm{GR}(T).$$

Inclusion can be strict! Take
$T = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 = 3x_1^2 x_2 = F$.

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{For GR solve } \begin{cases} x^T A_1 y = x_1 y_2 + x_2 y_1 = 0 \\ x^T A_2 y = x_1 y_1 = 0 \end{cases} \implies \mathrm{GR}(T) = 2.$$

$$\text{For SGR solve } \begin{cases} x^T A_1 x = 2x_1 x_2 = 0 \\ x^T A_2 x = x_1^2 = 0 \end{cases} \implies \mathrm{SGR}(T) = 1.$$

Reference: J Lindberg, P Santarsiero: *The symmetric geometric rank of symmetric tensors*. arXiv preprint, arXiv:2303.17537, 2023.

Questions?

Thank you for the attention!