Tensor rank and substitution method

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Plan

- Tensors, restrictions and tensor rank
- Direct sums and Kronecker products of tensors
- Properties of tensor rank. Asymptotic rank
- Lower bounds. Substitution method

Tensors: abstract definition

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- There is a bijective correspondence

$$\text{trilinear } F\colon U \times V \times W \to X \quad \leftrightarrow \quad \text{linear } L\colon U \otimes V \otimes W \to X$$

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Tensors: concrete representation

- Let $(u_1, \ldots, u_\ell)$, $(v_1, \ldots, v_m)$, $(w_1, \ldots, w_n)$ be bases of U, V, W
- Then $u_i \otimes v_j \otimes w_k$ form a basis of $U \otimes V \otimes W$

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- An order three tensor T is given by a three-way array (t_{ijk})

Restriction of tensors

- Let $A\colon U \to U'$, $B\colon V \to V'$, $C\colon W \to W'$ be linear maps
- Then we have a linear map

$$(A \otimes B \otimes C)\colon U \otimes V \otimes W \to U' \otimes V' \otimes W'$$

defined by the identity

$$(A \otimes B \otimes C)(u \otimes v \otimes w) = (Au \otimes Bv \otimes Cw)$$

Restriction of tensors

Definition (Restriction preorder)

T is a restriction of S if $T = (A \otimes B \otimes C)S$ for some linear maps A, B, C
Notation: $T \leq S$

Definition (Equivalence of tensors)

Tensors T and S are equivalent if $T \leq S$ and $S \leq T$.

Tensor rank

Definition (Diagonal tensor)

$$\mathsf{I}_r = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i \in \mathbb{F}^r \otimes \mathbb{F}^r \otimes \mathbb{F}^r$$

Definition (Tensor rank)

$$R(T) = \min\{r \mid T \leq \mathsf{I}_r\}$$

Example

$$I_3 = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 + e_3 \otimes e_3 \otimes e_3$$
$$W = e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0$$

Example

$$I_3 = |1\rangle \otimes |1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle \otimes |2\rangle + |3\rangle \otimes |3\rangle \otimes |3\rangle$$
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$$\mathsf{W} = |0\rangle \otimes |0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle \otimes |0\rangle$$

$$R(\mathsf{W}) \leq 3$$

$$\mathsf{W} = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right) \cdot \mathsf{I}_3$$

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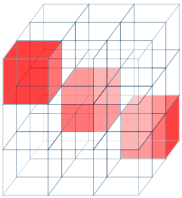

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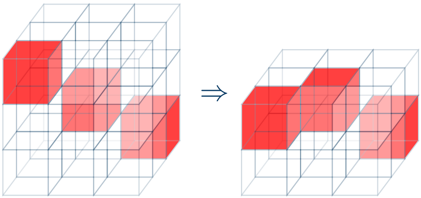

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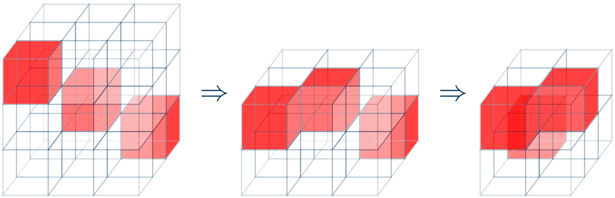

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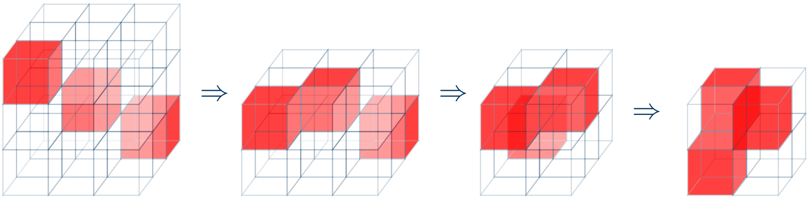

Rank decompositions

Definition

A decomposition of the form

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

is called a *rank decomposition* of T.

Theorem

$R(T)$ is the minimal number of summands in a rank decomposition of T.

- $R(T) \leq r \iff T \leq \mathsf{I}_r$

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- $R(T) \leq r \;\Leftrightarrow\; T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$ with $u_i{=}A\,|i\rangle, v_i{=}B\,|i\rangle, w_i{=}C\,|i\rangle$

Example

$$A_3 = |1\rangle \wedge |2\rangle \wedge |3\rangle = \sum_{\pi \in \mathfrak{S}_3} (-1)^\sigma \, |\pi(1)\rangle \otimes |\pi(2)\rangle \otimes |\pi(3)\rangle$$

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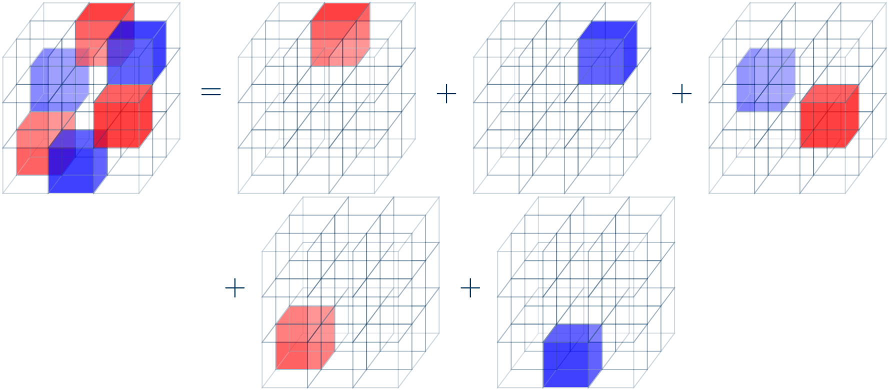

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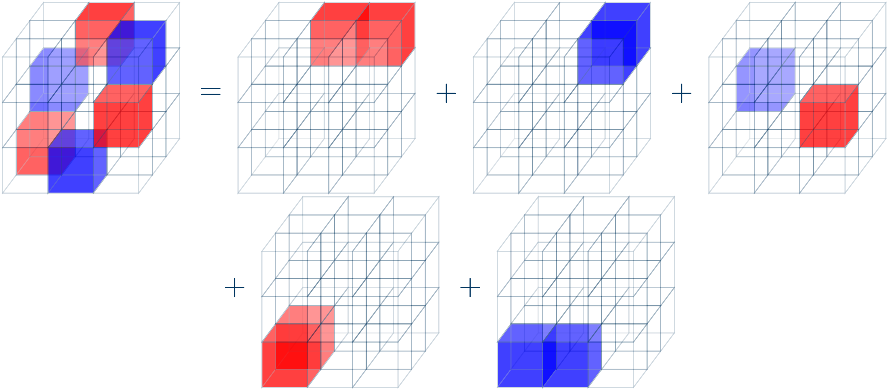

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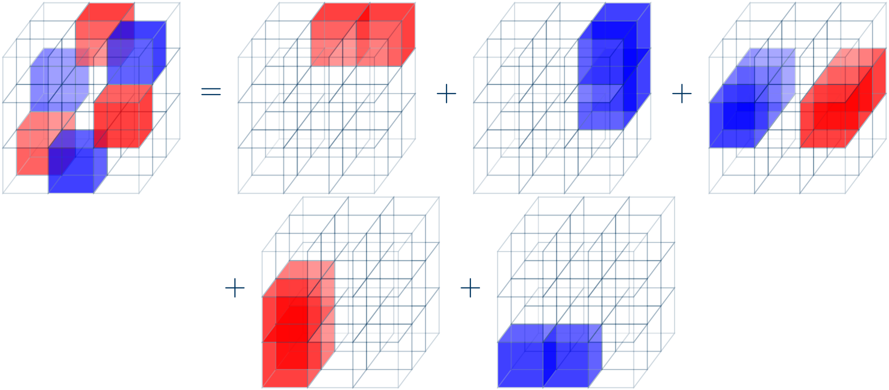

Properties of tensor rank

$$T \leq S \Rightarrow R(T) \leq R(S)$$

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$$R(\mathbf{I}_r) = r$$

- *I.O.U* a proof

Direct sum of tensors

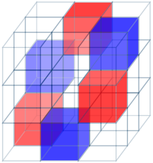
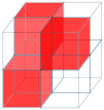
- Let U_1, V_1, W_1 and U_2, V_2, W_2 be vector spaces
- The injections $U_1 \hookrightarrow U_1 \oplus U_2$, $V_1 \hookrightarrow V_1 \oplus V_2$, $W_1 \hookrightarrow W_1 \oplus W_2$ give an injection $U_1 \otimes V_1 \otimes W_1 \hookrightarrow (U_1 \oplus U_2) \otimes (V_1 \oplus V_2) \otimes (W_1 \oplus W_2)$
- Same for $U_2 \otimes V_2 \otimes W_2$

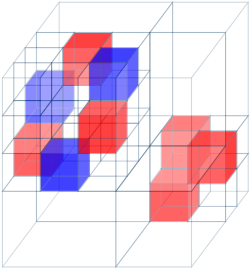
Definition (Direct sum)

For $T_1 \in U_1 \otimes V_1 \otimes W_1$ and $T_2 \in U_2 \otimes V_2 \otimes W_2$ their *direct sum* is the sum of their embeddings in $(U_1 \oplus U_2) \otimes (V_1 \oplus V_2) \otimes (W_1 \oplus W_2)$

Direct sum: example

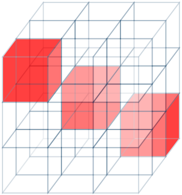
- The space $(U_1 \oplus U_2) \otimes (V_1 \oplus V_2) \otimes (W_1 \oplus W_2)$ decomposes into 8 "blocks" $U_i \otimes V_j \otimes W_k$.
- Direct sums use "diagonal blocks"

$A_3 =$  $W =$ 

$A_3 \oplus W =$ 

Direct sum: diagonal tensors

- We have seen this diagonal placement before

$I_3 =$ 

- Note $\mathbb{F}^a \oplus \mathbb{F}^b \cong \mathbb{F}^{a+b}$
- Using this isomorphism on all three factors, we get

$$I_a \oplus I_b \sim I_{a+b}$$

- This gives an alternative definition of diagonal tensors

$$I_1 = 1 \otimes 1 \otimes 1 \in \mathbb{F} \otimes \mathbb{F} \otimes \mathbb{F}; \qquad I_a = I_1^{\oplus a}$$

Direct sum and restrictions

$$((A_1 \oplus A_2) \otimes (B_1 \oplus B_2) \otimes (C_1 \oplus C_2))(T_1 \oplus T_2) = \\ = [(A_1 \otimes B_1 \otimes C_1)T_1] \oplus [(A_2 \otimes B_2 \otimes C_2)T_2]$$

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$$\begin{cases} T_1 \leq S_1 \\ T_2 \leq S_2 \end{cases} \Rightarrow T_1 \oplus T_2 \leq S_1 \oplus S_2$$

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$$\begin{cases} T_1 \leq S_1 \\ T_2 \leq S_2 \end{cases} \Rightarrow T_1 \oplus T_2 \leq S_1 \oplus S_2$$

$$R(T_1 \oplus T_2) \leq R(T_1) + R(T_2)$$

Direct sum conjecture

- Strassen conjectured that $R(T_1 \oplus T_2) = R(T_1) + R(T_2)$
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Direct sum conjecture

- Strassen conjectured that $R(T_1 \oplus T_2) = R(T_1) + R(T_2)$
- The conjecture was disproven in 2017 by Shitov
- The proof uses generic tensors of a special form
- No explicit pair of tensors with $R(T_1 \oplus T_2) < R(T_1) + R(T_2)$ is known

Tensor product and Kronecker product

- It will be useful for us to introduce two different tensor products $U \otimes V$ and $U \boxtimes V$
- Intuitively, we think of $U \otimes V \otimes \dots$ as matrices and tensors, and $U \boxtimes V \boxtimes \dots$ as "long vectors" composed of other vectors

$$(u_1, u_2, u_3) \otimes (v_1, v_2) = \begin{bmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \\ u_3 v_1 & u_3 v_2 \end{bmatrix}$$

$$(u_1, u_2, u_3) \boxtimes (v_1, v_2) = (u_1 v | u_2 v | u_3 v) = \\ = (u_1 v_1, u_1 v_2, u_2 v_1, u_2 v_2, u_3 v_1, u_3 v_2)$$

- Of course, $U \otimes V$ and $U \boxtimes V$ are isomorphic as vector spaces and tensor products, the difference is purely syntactic convenience

Kronecker product of tensors

Definition (Kronecker product)

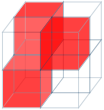
For $T_1 \in U_1 \otimes V_1 \otimes W_1$ and $T_2 \in U_2 \otimes V_2 \otimes W_2$ we define the *Kronecker product* $T_1 \boxtimes T_2 \in (U_1 \boxtimes U_2) \otimes (V_1 \boxtimes V_2) \otimes (W_1 \boxtimes W_2)$ as a bilinear function of T_1 and T_2 satisfying

$$(u_1 \otimes v_1 \otimes w_1) \boxtimes (u_2 \otimes v_2 \otimes w_2) = (u_1 \boxtimes u_2) \otimes (v_1 \boxtimes v_2) \otimes (w_1 \boxtimes w_2)$$

- Let $T = (t_{ijk})$. Then $T \boxtimes S = \sum_{i,j,k} (|i\rangle \otimes |j\rangle \otimes |k\rangle) \boxtimes (t_{ijk} S)$
- $T \boxtimes S$ has the "outer structure" of T, but instead of scalars, we have scalar multiples of S as blocks

Kronecker product: example

$$W = |0\rangle \otimes |0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle \otimes |0\rangle$$

$W =$ 

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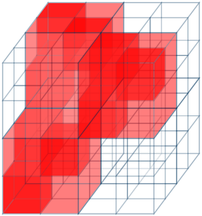
$W \boxtimes W =$ 

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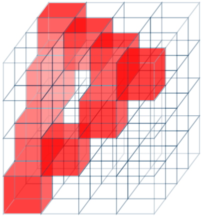
$W \boxtimes W =$ 

Matrix multiplication tensors

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$$\mathsf{M}_{a11} \sim \sum_{i=1}^{a} |i\rangle \otimes 1 \otimes |i\rangle \in \mathbb{F}^a \otimes \mathbb{F} \otimes \mathbb{F}^a$$

$$\mathsf{M}_{1b1} \sim \sum_{j=1}^{b} |j\rangle \otimes |j\rangle \otimes 1 \in \mathbb{F}^b \otimes \mathbb{F}^b \otimes \mathbb{F}$$

$$\mathsf{M}_{11c} \sim \sum_{k=1}^{c} 1 \otimes |k\rangle \otimes |k\rangle \in \mathbb{F} \otimes \mathbb{F}^c \otimes \mathbb{F}^c$$

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$$\mathsf{M}_{abc} \sim \mathsf{M}_{a11} \boxtimes \mathsf{M}_{1b1} \boxtimes \mathsf{M}_{11c}$$

Kronecker product and restrictions

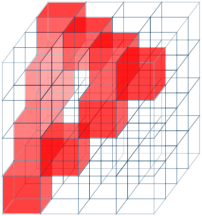
$$\mathrm{I}_a \boxtimes \mathrm{I}_b = \mathrm{I}_{ab}$$

$$((A_1 \boxtimes A_2) \otimes (B_1 \boxtimes B_2) \otimes (C_1 \boxtimes C_2)) \cdot (T_1 \boxtimes T_2) =$$
$$= ((A_1 \otimes B_1 \otimes C_1) T_1) \boxtimes ((A_2 \otimes B_2 \otimes C_2) T_2)$$

$$\begin{cases} T_1 \leq S_1 \\ T_2 \leq S_2 \end{cases} \Rightarrow T_1 \boxtimes T_2 \leq S_1 \boxtimes S_2$$

$$R(T_1 \boxtimes T_2) \leq R(T_1) R(T_2)$$

$R(\mathrm{W} \boxtimes \mathrm{W}) \leq 7$

$\mathrm{W} \boxtimes \mathrm{W} =$ 

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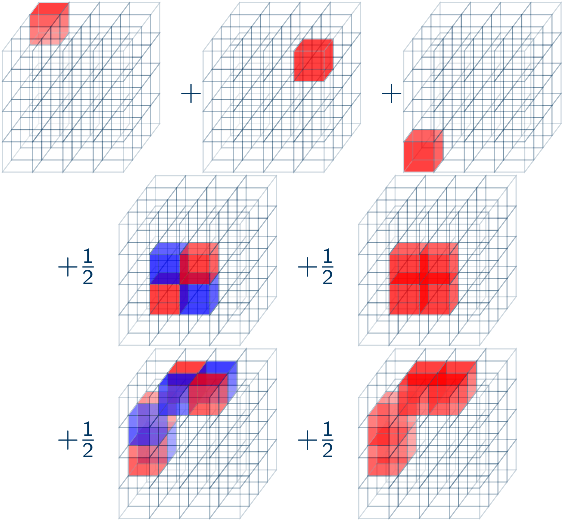

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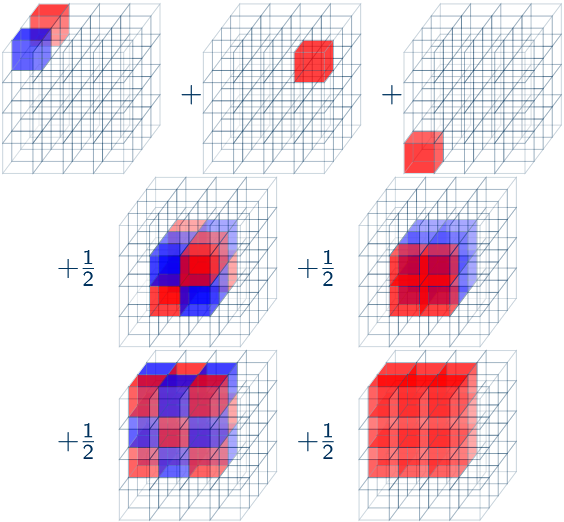

Properties of rank

Monotonicity:	$T \leq S \Rightarrow R(T) \leq R(S)$
Subadditivity:	$R(T \oplus S) \leq R(T) + R(S)$
Submultiplicativity:	$R(T \boxtimes S) \leq R(T) \cdot R(S)$
Normalization:	$R(\mathsf{I}_r) = r$

Asymptotic behaviour of rank

- Denote $\rho_n = \log R(T^{\boxtimes n})$
- From the properties of rank it follows that ρ is subaddtive

$$\rho_{n+m} \leq \rho_n + \rho_m$$

- Fekete's lemma: $\frac{\rho_n}{n}$ converges

Definition

Asymptotic rank of T is defined as

$$\underset{\sim}{R}(T) = \lim_{n \to \infty} \left(R(T^{\boxtimes n}) \right)^{\frac{1}{n}}$$

- We have $R(T^{\boxtimes n}) = \underset{\sim}{R}(T)^{n+o(n)}$

Properties of MM tensors

- As for the diagonal tensors, we have

$$M_{a11} \boxtimes M_{a'11} = M_{aa',1,1}$$

- And the same for M_{1b1} and M_{11b}

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- As for the diagonal tensors, we have

$$\mathsf{M}_{a11} \boxtimes \mathsf{M}_{a'11} = \mathsf{M}_{aa',1,1}$$

- And the same for M_{1b1} and M_{11b}
- It follows that

$$\mathsf{M}_{abc} \boxtimes \mathsf{M}_{a'b'c'} = \mathsf{M}_{aa',bb',cc'}$$

- This property has interpretation for matrix multiplication maps: block matrices can be multiplied blockwise

Matrix multiplication complexity

- The question about asymptotic complexity of matrix multiplication can be stated in terms of tensor rank

$$R(\mathsf{M}_{nnn}) = n^{\omega + o(1)}$$

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$$R(\mathsf{M}_{nnn}) = n^{\omega + o(1)}$$

- Note that $\mathsf{M}_{aaa} \boxtimes \mathsf{M}_{bbb} = \mathsf{M}_{ab,ab,ab}$

$$\underset{\sim}{R}(\mathsf{M}_{222}) = \lim \sqrt[n]{R(\mathsf{M}_{2^n,2^n,2^n})} = \lim \sqrt[n]{2^{\omega n + o(n)}} = 2^{\omega}$$

Upper and lower bounds

- In applications, tensor rank provides a measure of complexity for objects represented by tensors

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- Algebraic Computation: complexity of computing multilinear maps
- Upper bounds $\approx$ “algorithms”, lower bounds $\approx$ “hardness proofs”
- Restrictions $\approx$ “reductions between problems”

Upper and lower bounds

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- Algebraic Computation: complexity of computing multilinear maps
- Upper bounds $\approx$ "algorithms", lower bounds $\approx$ "hardness proofs"
- Restrictions $\approx$ "reductions between problems"

- Computing tensor rank is hard; the exact value is known only for small or very restricted tensors
- Have some construction for upper bounds / explicit restrictions
- The situation with lower bounds is much worse

Tensor rank and ranks of tensors

- There are other notions of rank for tensors
- Subrank, slice rank, flattening rank ...
- What are the common properties of these ranks?

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Definition

$F\colon \{\text{tensors}\} \to \mathbb{R}_+$ is a *rank functional* if it satisfies

$$\begin{aligned} \text{Monotonicity:} \quad & T \leq S \Rightarrow F(T) \leq F(S) \\ \text{Normalization:} \quad & F(\mathsf{I}_r) = r \end{aligned}$$

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Proposition

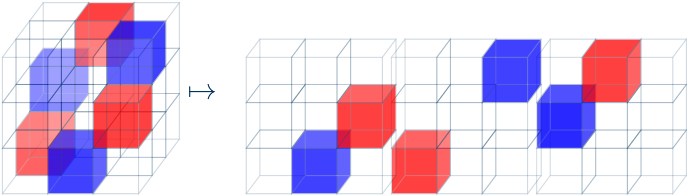
If there exists some rank functional, then $R(\mathsf{I}_r) = r$

- *Proof*: $R(\mathsf{I}_r) \ge F(\mathsf{I}_r) = r$. It is obvious that $R(\mathsf{I}_r) \le r$.

Flattening and flattening rank

- Flattening is a way to transform a tensor into a matrix
- Flattening with respect to the 1st factor

$$T \in U \otimes V \otimes W \;\mapsto\; \mathcal{F}_1(T) \in U \otimes (V \boxtimes W)$$
$$u \otimes v \otimes w \;\mapsto\; u \otimes (v \boxtimes w)$$



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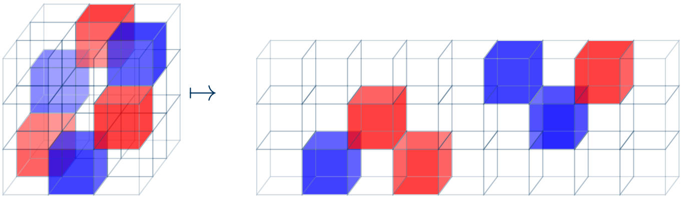

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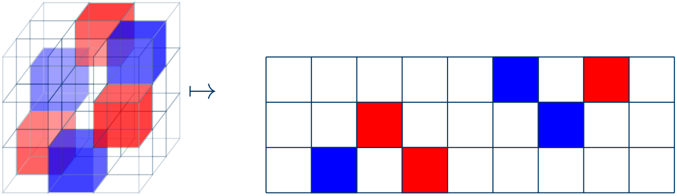

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Flattening rank of T is the rank of the flattening

$$R_1(T) = \operatorname{rk} \mathcal{F}_1(T)$$

Proposition

Flattening rank is a rank functional

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  Or, in terms of matrix multiplication, $\mathcal{F}_1(T) = A \cdot \mathcal{F}_1(S) \cdot (B \boxtimes C)^\top$

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- Known lower bound methods
  - Continuous methods
  - Substitution method
  - Coding theory methods over finite fields

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Definition (Conciseness)

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Proposition

If $T = (A \otimes B \otimes C)S$ and T is 1-concise, then A is surjective

- *Proof:* As matrices $\mathcal{F}_1(T) = A \cdot \mathcal{F}_1(S) \cdot (B \boxtimes C)^\top$
  Therefore $\operatorname{rk} A \geq \operatorname{rk} \mathcal{F}_1(T) = \dim U$

Corollary

If $T = \sum_i u_i \otimes v_i \otimes w_i$ and T is 1-concise, then $\{u_i\}$ generates U.

Substitution method

Theorem (Substitution method)

Let $T \in U \otimes V \otimes W$ be a 1-concise tensor, and $X \subset U$.
Then there exists a projection $\Pi \colon U \to X$ such that

$$R(T) \geq R\left((\Pi \otimes \mathrm{Id} \otimes \mathrm{Id})T\right) + \Delta$$

where $\Delta = \dim U - \dim X$

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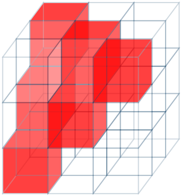
- $(\Pi \otimes \mathrm{Id} \otimes \mathrm{Id}) \cdot T = \sum_{i=\Delta+1}^{r} \Pi u_i \otimes v_i \otimes w_i$
- Therefore $R((\Pi \otimes \mathrm{Id} \otimes \mathrm{Id}) \cdot T) \leq r - \Delta$

Example

$$\mathrm{P}_n = \sum_{i+j+k=n-1} |i\rangle \otimes |j\rangle \otimes |k\rangle \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$$



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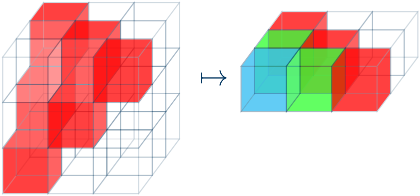

- Let $\Pi : \mathbb{F}^n \to \mathrm{Span}(|0\rangle)$ be a projection. $\Pi |i\rangle = \alpha_i |0\rangle$, with $\alpha_0 = 1$

$$(\Pi \otimes \mathrm{Id} \otimes \mathrm{Id}) \cdot \mathrm{P}_n = \sum_{i+j+k=n-1} \Pi |i\rangle \otimes |j\rangle \otimes |k\rangle = \sum_{j+k=n-1-i} \alpha_i |0\rangle \otimes |j\rangle \otimes |k\rangle$$

- $(\Pi \otimes \mathrm{Id} \otimes \mathrm{Id}) \cdot \mathrm{P}_n = |0\rangle \otimes M$ where M is a triangular matrix with 1 on the diagonal
- $R(M) = n \;\Rightarrow\; R(\mathrm{P}_n) \geq n + (n-1) = 2n - 1$

Generalization of the example: attempt 1

Definition (Contraction)

For $f \in U^*$, denote $Tf = (f \otimes \mathrm{Id} \otimes \mathrm{Id}) \cdot T \in \mathbb{F} \otimes V \otimes W \cong V \otimes W$

Definition (Minrank)

Define *minrank* of T as

$$mr(T) = \min\{\mathrm{rk}(Tf) \mid f \neq 0\}$$

Theorem

For a 1-concise tensor $T \in U \otimes V \otimes W$

$$R(T) \geq mr(T) + \dim U - 1$$

- Does not generalize the example: $mr(\mathrm{P}_n) = 1$

Questions?

Thank you!