Grand Unification of Quantum Algorithms

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Quantum algorithm design

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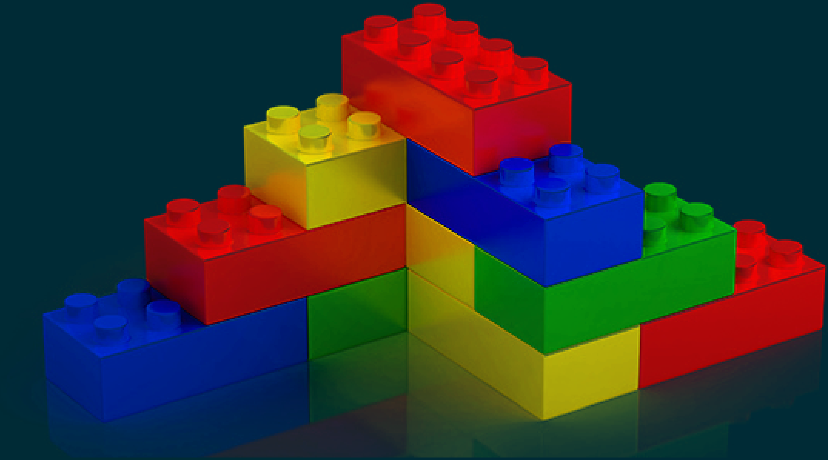

Many quantum algorithms have a common structure!

A bird's eye view on quantum linear algebra

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Motivating example - the quantum matrix inversion (HHL) algorithm

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Matrix arithmetic on a quantum computer using block-encoding

$$\text{Input matrix}: \ A; \ \text{Implementation}: \ U = \begin{bmatrix} A & \cdot \\ \cdot & \cdot \end{bmatrix}; \ \text{Algorithm}: \ U' = \begin{bmatrix} f(A) & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

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More examples

- *Optimal Hamiltonian simulation [Low et al.], quantum walks [Szegedy]*
- *Fixed point [Yoder et al.] and oblivious amplitude amplification [Berry et al.]*
- *HHL, regression [Chakraborty et al.], SDPs & LPs [Brandão et al.], ML [Kerendis et al.]*

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A way to represent large matrices on a quantum computer efficiently

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$$U = \begin{bmatrix} A & \cdot \\ \cdot & \cdot \end{bmatrix} \iff A = (\langle 0|^a \otimes I)\, U\, (|0\rangle^a \otimes I)\,.$$

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- a POVM operator M given we can sample from the rand.var.: $\mathrm{Tr}(\rho M)$,

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Then $(P^\dagger \otimes I)U(P \otimes I)$ is a block-encoding of $\sum_i p_i U_i$.
In particular if $(\langle 0| \otimes I)U_i(|0\rangle \otimes I) = A_i$, then it is a block-encoding of

$$\sum_i p_i A_i.$$

Quantum Singular Value Transformation (QSVT)

Our main theorem about QSVT

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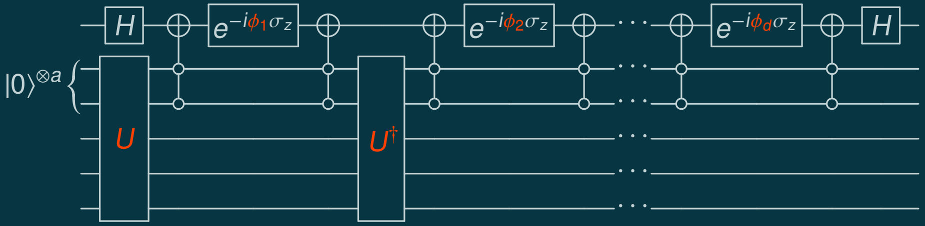

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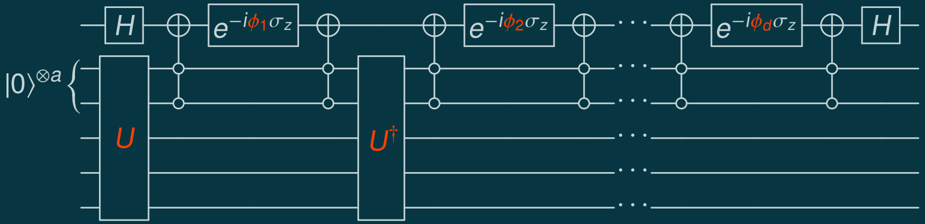

Simmilar result holds for even polynomials.

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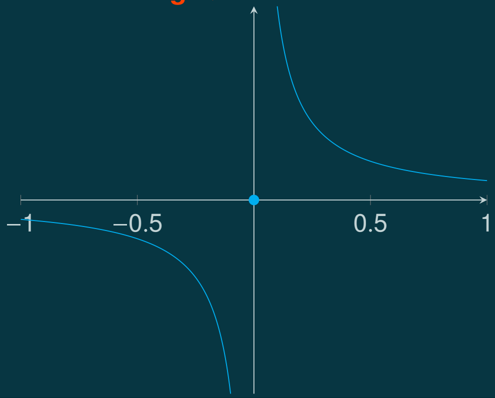

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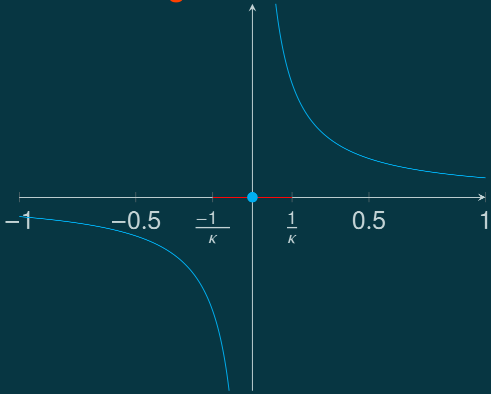

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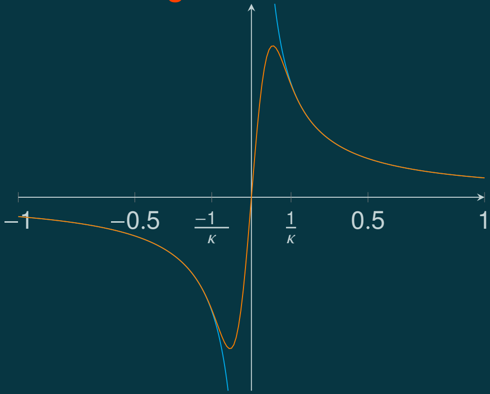

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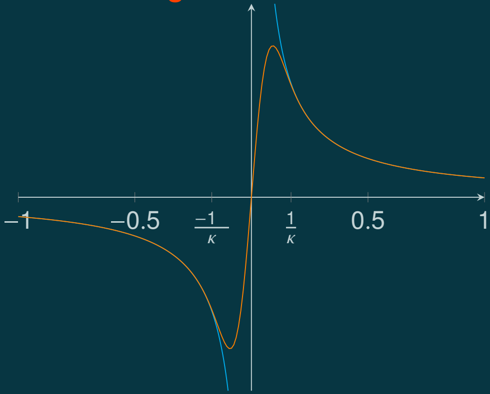

Degree / complexity: $\mathcal{O}\left(\kappa \log\left(\frac{1}{\varepsilon}\right)\right)$

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$$U' = \begin{bmatrix} M^t & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

Proof: x^t can be ε-apx. on $[-1, 1]$ with a degree-$\sqrt{2t \ln(2/\varepsilon)}$ polynomial.

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$$U' = \begin{bmatrix} P(H) & . \\ . & . \end{bmatrix},$$

using d times U and $U^\dagger$, 1 controlled U, and $\mathcal{O}(ad)$ extra two-qubit gates.

Proof: let $P_{\text{even}}(x) := P(x) + P(-x)$ and $P_{\text{odd}}(x) := P(x) - P(-x)$ then

$$P(H) = \frac{1}{2}(P_{\text{even}}(H) + P_{\text{odd}}(H)) \quad \text{implement using QSVT + LCU}$$

Quantum signal processing & proof sketch of QSVT

Single qubit quantum control using σ_z phases?

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$$R(x) := \begin{bmatrix} x & -\sqrt{1-x^2} \\ -\sqrt{1-x^2} & -x \end{bmatrix}; \quad e^{i\phi_0\sigma_z}R(x)e^{i\phi_1\sigma_z}\cdot\ldots\cdot R(x)e^{i\phi_d\sigma_z} = (*)?$$

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Theorem: Basic characterization [Low, Yoder, Chuang (2016)]

Let $d \in \mathbb{N}$; for all $\Phi \in \mathbb{R}^{d+1}$ we have

$$(*) = i^d \begin{bmatrix} P_{\mathbb{C}}(x) & Q_{\mathbb{C}}(x) i\sqrt{1-x^2} \\ Q^*_{\mathbb{C}}(x) i\sqrt{1-x^2} & P^*_{\mathbb{C}}(x) \end{bmatrix},$$

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(iii) $\forall x \in [-1,1]$: $|P_{\mathbb{C}}(x)|^2 + (1-x^2)|Q_{\mathbb{C}}(x)|^2 = 1$.

Real quantum signal processing

Theorem: Focusing on the real part [Low, Yoder, Chuang (2016)]

Let $d \in \mathbb{N}$, and $P \in \mathbb{R}[x]$ be of degree d. There exists $\Phi \in \mathbb{R}^d$ such that

$$\prod_{j=1}^{d} \left(R(x) e^{i\phi_j \sigma_z} \right) = \begin{bmatrix} P_{\mathbb{C}}(x) & \cdot \\ \cdot & \cdot \end{bmatrix},$$

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Implementing the real part of a polynomial map

Direct implementation

$$-\boxed{e^{i\phi_d\sigma_z}}-\boxed{R(x)}-\boxed{e^{i\phi_{d-1}\sigma_z}}-\cdots-\boxed{R(x)}-\boxed{e^{i\phi_0\sigma_z}}- = \begin{bmatrix} P_{\mathbb{C}}(x) & . \\ . & . \end{bmatrix}$$

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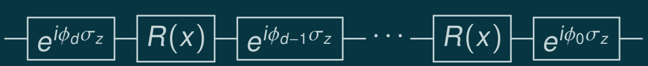

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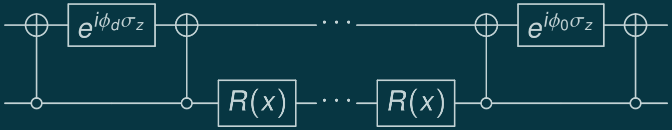

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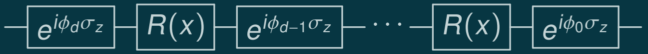

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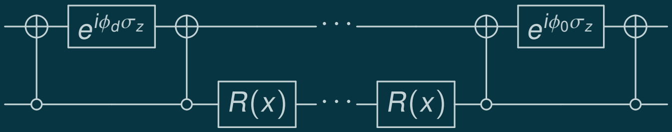

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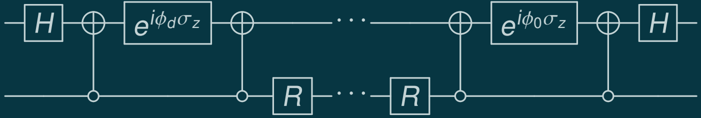

$$= \begin{bmatrix} \Re[P_{\mathbb{C}}] & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Generalisation to higher dimensions

1×1 case

$$\text{Input:} \begin{bmatrix} x & \cdot \\ \cdot & \cdot \end{bmatrix} \quad \text{Modulation:} \begin{bmatrix} e^{i\phi} & \\ & e^{-i\phi} \end{bmatrix} \quad \text{Output:} \begin{bmatrix} P(x) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

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2×2 case (higher-dimensional case is similar)

Input unitary	Modulation	Output circuit
$\begin{bmatrix} x & \cdot & & \\ \cdot & \cdot & & \\ & & y & \cdot \\ & & \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} e^{i\phi} & & & \\ & e^{-i\phi} & & \\ & & e^{i\phi} & \\ & & & e^{-i\phi} \end{bmatrix}$	$\begin{bmatrix} P(x) & \cdot & & \\ \cdot & \cdot & & \\ & & P(y) & \cdot \\ & & \cdot & \cdot \end{bmatrix}$

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Fast QMA gap amplification [Marriott-Watrous'05] [Nagaj et al.'09]

The language class QMA

The language L belongs to the class QMA if for every input length $|x|$ there exists a quantum verifier $V_{|x|}$, and numbers $0 \leq b_{|x|} < a_{|x|} \leq 1$ satisfying $\frac{1}{a_{|x|}-b_{|x|}} = \mathcal{O}\left(\mathrm{poly}\left(|x|\right)\right)$, such that for all

$x \in L$ there exists a witness $|\psi\rangle$ such that upon measuring the state $V_{|x|}|x\rangle|0\rangle^m|\psi\rangle$ the probability of finding the $(|x|+1)$st qubit in state $|1\rangle$ has probability at least $a_{|x|}$,

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We can modify the verifier circuit $V_{|x|}$ such that the acceptance probability thresholds become $a' := 1-\varepsilon$ and $b' := \varepsilon$ using singular value transformation of degree $\mathcal{O}\left(\frac{1}{\sqrt{a_{|x|}}-\sqrt{b_{|x|}}}\log\left(\frac{1}{\varepsilon}\right)\right)$.

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Observe that by the above definition

$$\forall x \in L: \left\|\left(\langle x|\otimes|1\rangle\langle 1|\otimes I_{n+m-1}\right) V\left(|x\rangle\otimes|0\rangle\langle 0|^{\otimes m}\otimes I_n\right)\right\| \geq \sqrt{a_{|x|}},$$

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Singular vector transformation and projection

Fixed-point and oblivious amplitude ampl. [Yoder et al., Berry et al.]

Amplitude amplification problem: Given U such that

$$U|0\rangle = \sqrt{p}|0\rangle\left|\psi_{\text{good}}\right\rangle + \sqrt{1-p}|1\rangle|\psi_{\text{bad}}\rangle, \quad \text{prepare } \left|\psi_{\text{good}}\right\rangle.$$

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Note that $(|0\rangle\langle 0| \otimes I)U(|0\rangle\langle 0|) = \sqrt{p}|0, \psi_{\text{good}}\rangle\langle 0|$; we can apply QSVT.

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Generalization: Singular vector transformation

Given a unitary U, and projectors $\widetilde{\Pi}, \Pi$, such that

$$A = \widetilde{\Pi} U \Pi = \sum_{i=1}^{k} \varsigma_i |\phi_i\rangle\langle\psi_i|$$

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If $\varsigma_i \geq \delta$ for all $0 \neq \alpha_i$, we can ε-apx. using QSVT with compl. $\mathcal{O}\left(\frac{1}{\delta}\log\left(\frac{1}{\varepsilon}\right)\right)$.

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Given $t, \varepsilon > 0$, implement a unitary U', which is ε close to e^{itH}. Can be achieved with query complexity

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Proof sketch

Approximate to ε-precision $\sin(tx)$ and $\cos(tx)$ with polynomials of degree as above. Then use QSVT and combine even/odd parts.

Optimal complexity

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The same technique works for density operators!

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- Estimate $\tilde{p}_i$
- Output $f(\tilde{p}_i)$ E.g., for entropy output $-\log(\tilde{p}_i)$
- Estimate $\mathbb{E}[f(\tilde{p}_i)]$ by repeating the process

Quantum improvement: use amplitude estimation
(Bravyi, Harrow and Hassidim – 2009)

Suppose we can implement "quantum sampling": $U_p : |0\rangle \mapsto \sum_i \sqrt{p_i}|\phi_i\rangle|i\rangle$
Observation: a block encoding of $\sum_i \sqrt{p_i}|\tilde{\phi}_i\rangle\langle i|$ suffices and can be constructed!

The same technique works for density operators!
Purified access $U_\rho : |0\rangle \mapsto \sum_i \sqrt{p_i}|\phi_i\rangle|\psi_i\rangle$, where $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$

Trick - skip estimating p_i

Operationally access and transform the probabilities:

$$U_p$$

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Apply the operation to a sample:

$$U'_{f(p)}|0\rangle \sum_{i=1} \sqrt{p_i}|i\rangle|\psi_i\rangle = |0\rangle \sum_{i=1} \sqrt{p_i}\sqrt{f(p_i)}|i\rangle\left|\tilde{\psi}_i\right\rangle + |1\rangle \ldots$$

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Estimate the probability of measuring $|0\rangle$:

$$\sum_{i=1} p_i f(p_i) = \mathbb{E}[f(p)]$$

An intuitive lower bound

Lower bound on eigenvalue transformation

Suppose that U is a block-encoding of a Hermitian matrix H from a family of operators. Let $f\colon [-1,1] \to \mathbb{C}$, then implementing a block-encoding of $f(H)$ requires at least $\left\|\frac{df}{dx}\right\|_I$ uses of U, if $I \subseteq [-\frac{1}{2}, \frac{1}{2}]$ is an interval of potential eigenvalues of H.

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Proof sketch

The proof is based on an elementary argument about distinguishability of unitary operators.

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Proof sketch

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Optimality of pseudoinverse implementation

$$\text{Let } I := \left[\frac{1}{\kappa}, \frac{1}{2}\right] \text{ and let } f(x) := \frac{1}{\kappa x}, \text{ then } \left.\frac{df}{dx}\right|_{\frac{1}{\kappa}} = -\kappa.$$

Thus our implementation is optimal up to the $\log(1/\varepsilon)$ factor.

Summarizing the various speed-ups

Speed-up	Source of speed-up	Examples of algorithms
Exponential	Dimensionality of the Hilbert space	Hamiltonian simulation
	Precise polynomial approximations	Improved HHL algorithm
Quadratic	Singular value = square root of probability	Grover search
	Singular values are easier to distinguish	Amplitude estimation
	Close-to-1 singular values are more flexible	Quantum walks

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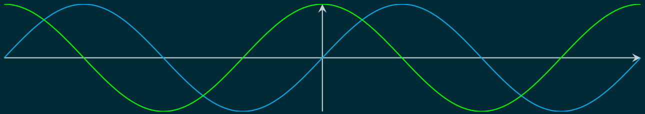
Some more applications

- Quantum walks, fast QMA amplification, fast quantum OR lemma
- Quantum Machine learning: PCA, principal component regression
- “Non-commutative measurements” (for ground state preparation)
- Sample and gate efficient metrology, fractional queries
- $\vdots$

Summary of some applications of QSVT

$\sin(tx), \cos(tx):$  Hamiltonian simulation

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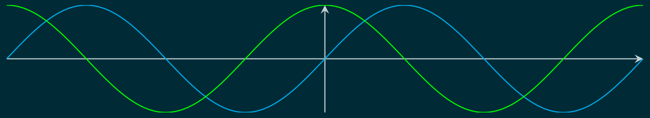
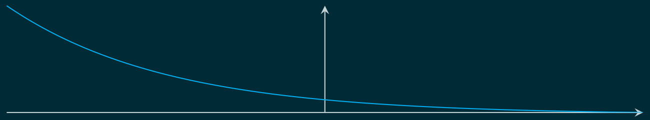


Hamiltonian simulation

$\exp(-\beta x):$



Gibbs sampling

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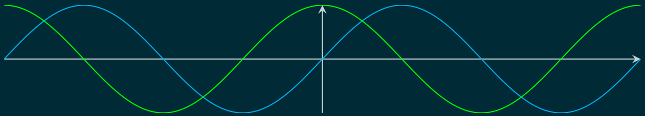

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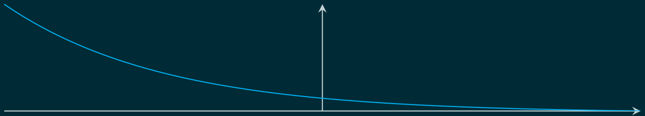

Gibbs sampling

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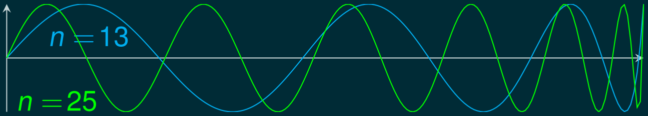

Grover search
Ampl. ampl.
Quantum walks

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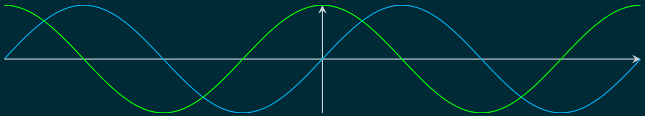

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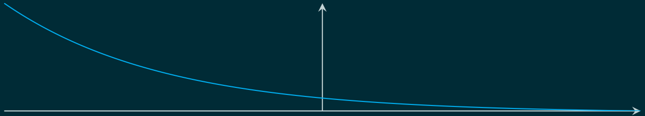

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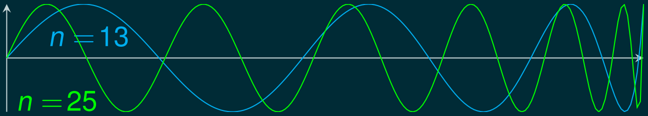

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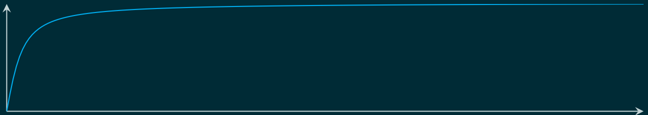
$\approx \text{Heaviside}(x):$



“Fixed-point” ampl. ampl.
Ground state prep.