Quantum machine learning

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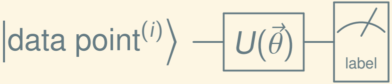
  –For example classification (supervised learning)



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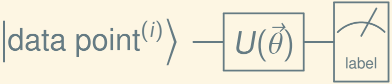

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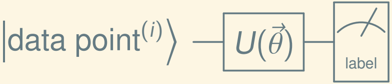

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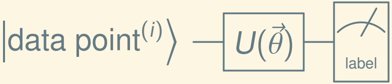

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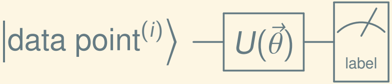

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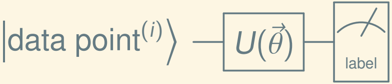

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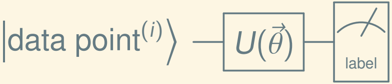

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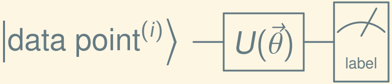

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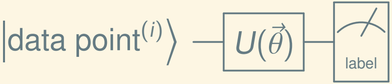

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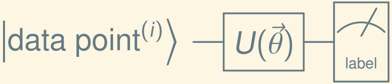

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- Learning from quantum data
  –Understanding properties of a quantum state or a quantum process

Quantum machine learning for Big Data

Some major tasks, given data $A \in \mathbb{R}^{m \times n}$

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- Need a circuit implementation (block-encoding) of the input matrix A
- Need to efficiently extract "answer" from the output $|x\rangle (= A^{-1}|b\rangle)$

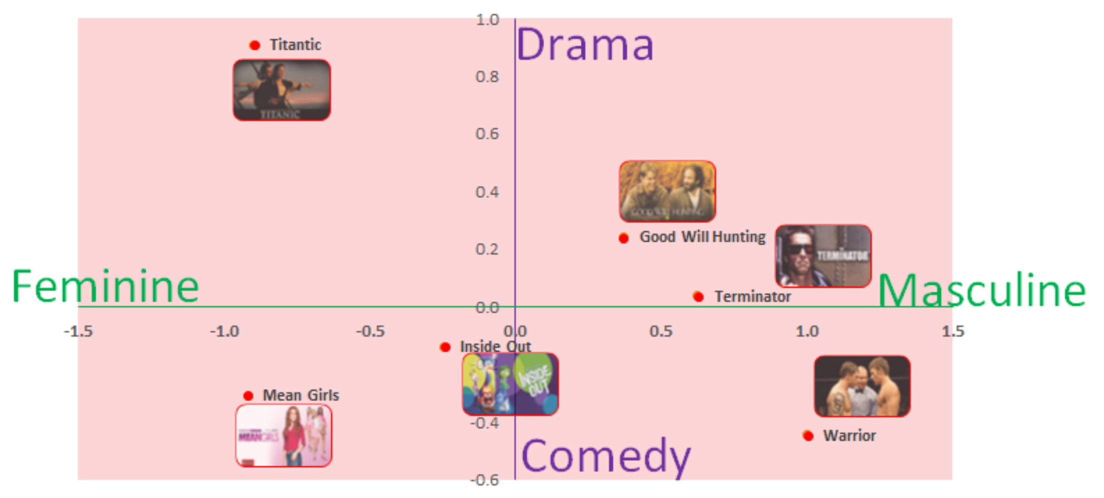
Recommendation systems – Netflix challange

	Inside Out	Good Will Hunting	Mean Girls	Terminator	Titanic	Warrior
Tina Fey	3	1	5	1	?	1
Helen Mirren	2	?	?	2	5	1
Sylvester Stallone	1	3	1	4	2	5
Tom Hanks	?	3	1	?	4	3
George Clooney	2	2	1	3	1	4

Image source: https://towardsdatascience.com ©

The assumed structure of preference matrix:

Movies: a linear combination of a small number of features
User taste: a linear weighing of the features



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Singular value decomposition

For every $A \in \mathbb{C}^{m \times n}$ its *singular value decomposition* is $A = U^{\dagger} \Sigma V$ where $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$ unitaries and $\Sigma \in \mathbb{R}^{m \times n}$ has non-zero elements only on the diagonal.

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We can also write $A = \sum_{i=1}^{m} \sigma_i |u_i\rangle\langle v_i|$, where u_i, v_i are the columns of U, V and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$ are the singular values of A.

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Fact: the best rank-k approximation of A is $\tilde{A} = \sum_{i=1}^{k} \sigma_i |u_i\rangle\langle v_i|$.
(Best in terms of the Frobenius norm: $\|M\|_F = \sqrt{\sum_{i,j} |M_{ij}|^2}$.)

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Measuring the state then gives recommendation j with probability $\propto |\tilde{A}_{ij}|^2$.

Major difficulty: how to input the data?

Data conversion: classical to quantum

- Given $b \in \mathbb{R}^m$ prepare

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- Given $A \in \mathbb{R}^{m \times n}$ construct quantum circuit (block-encoding)

$$U = \begin{pmatrix} A / \, ||A||_F & \cdot \\ \cdot & \cdot \end{pmatrix}.$$

How to preserve the exponential advantage?

Solution: assume QRAM (readable in superposition)

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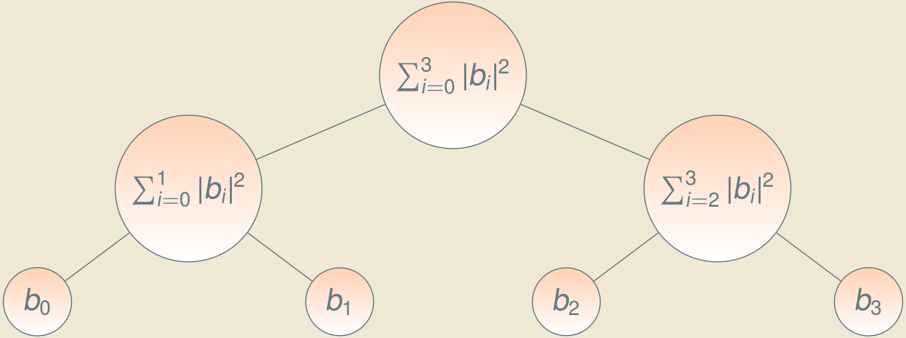
Data structure (for simplicity let us assume $\|b\| = 1$)



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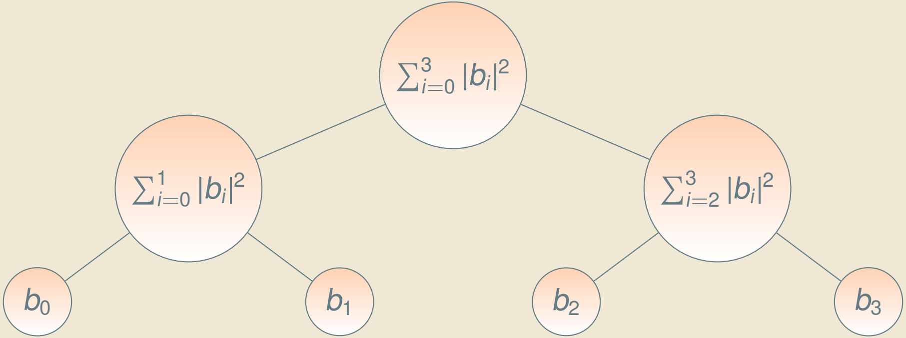

First prepare: $\sqrt{\sum_{i=0}^{1} |b_i|^2}|0\rangle + \sqrt{\sum_{i=2}^{3} |b_i|^2}|1\rangle$

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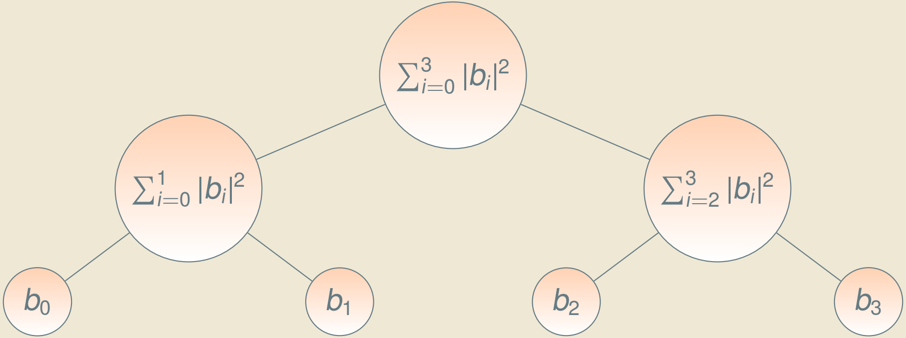

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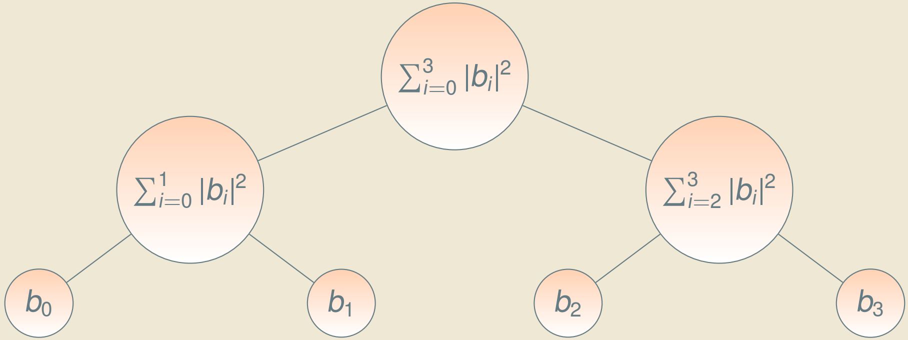

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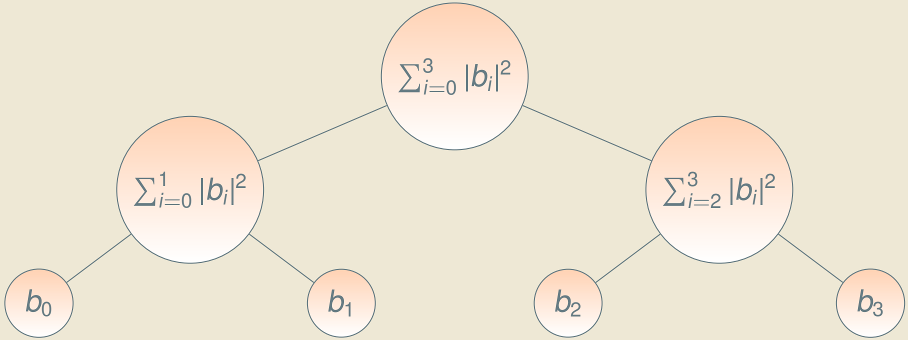

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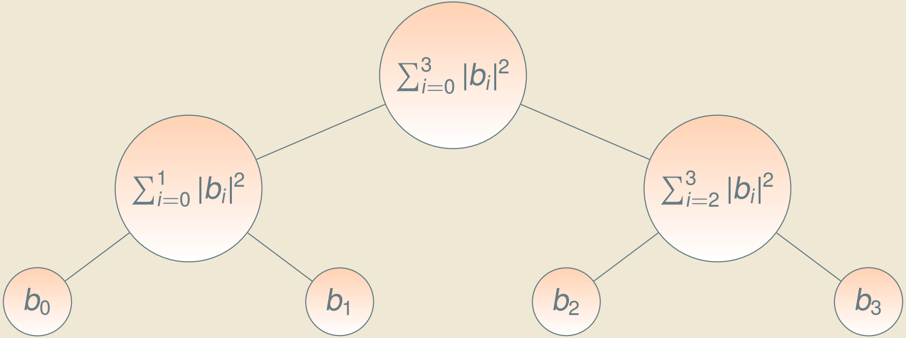

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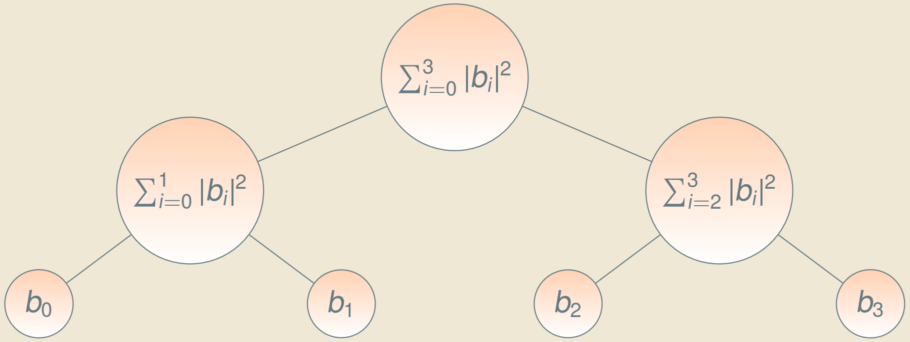
Add phases to get $b_0|00\rangle + b_1|01\rangle + b_2|10\rangle + b_3|11\rangle$

On-line updates to the data structure

Data structure



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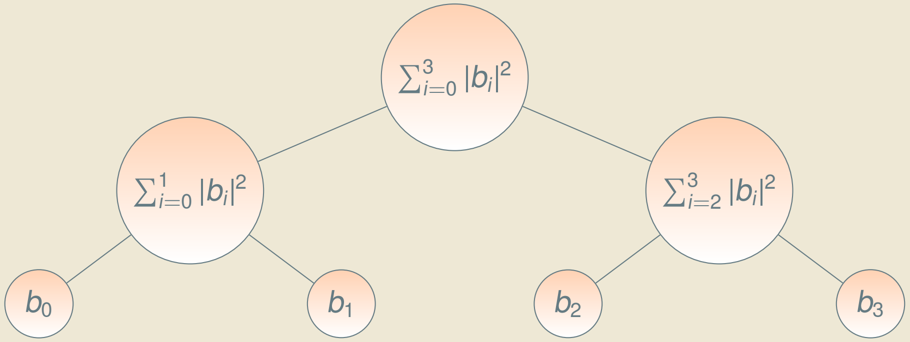

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Cost is about the depth: $\log(\text{dimension})$

Data structure for the matrix A

Let a be the vector of row norms such that $a_i = \|A_{i.}\|$.

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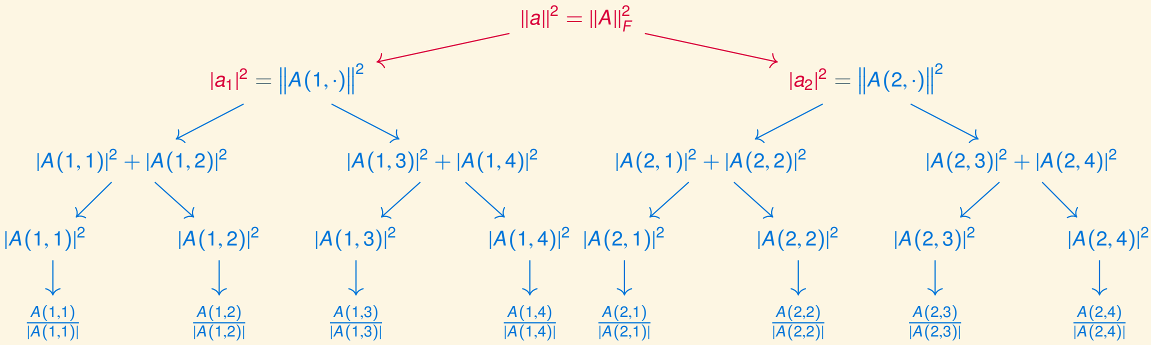

Dynamic data structure for a matrix $A \in \mathbb{C}^{2\times 4}$. We compose the data structure for a with the data structure for A's rows.

Quantum algorithms

Exercise 2: Let $R : |0\rangle|i\rangle \mapsto \frac{|A_{i.}\rangle|i\rangle}{\|A_{i.}\|}$ and $C : |0\rangle|j\rangle \mapsto \frac{|j\rangle|a\rangle}{\|a\|}$.
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Tomorrow we will see

This can be improved quadratically!

Surely exponential speed-up compared to classical, right?

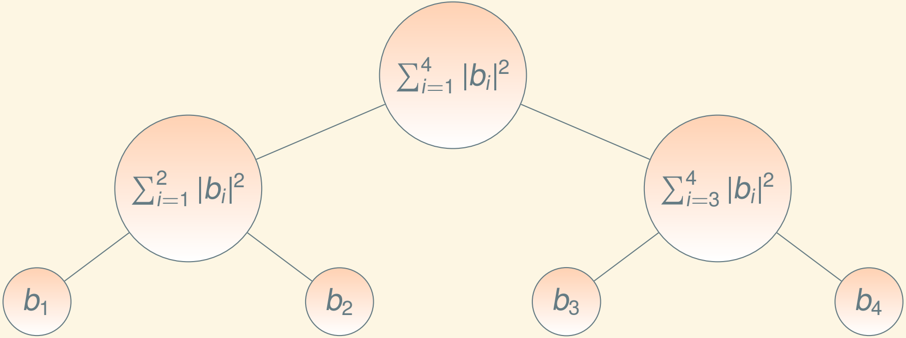
2018:

Ewin Tang
Quantum Recommendation Systems
arXiv:1807.04271
Ewin Tang
arXiv:1811.00414
Quantum PCA
Ewin Tang
arXiv:1811.04909
Low rank HHL

2018:

Image source: Quantum Computing Memes for QMA-Complete Teens

Sampling form the input vectors?



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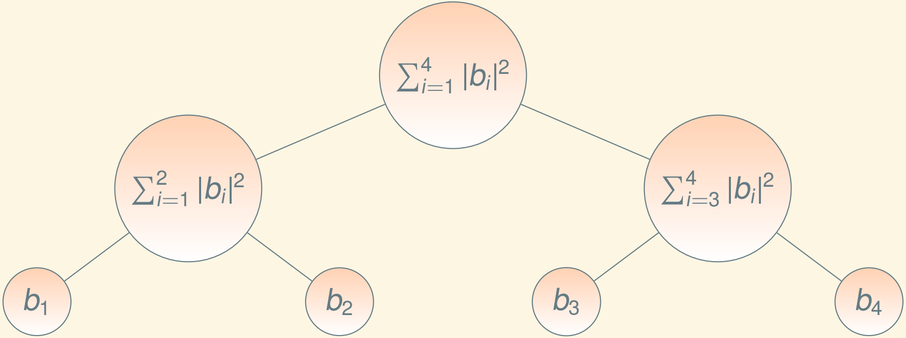

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Computing inner products

Computing $\langle x, y \rangle$ for normalized vectors x, y

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$A^\dagger A = \sum_{i=1}^{m} |A_{i.}\rangle\langle A_{i.}|$

With probability $\frac{\|A_{i.}\|^2}{\|A\|_F^2} = \frac{|a_i|^2}{\|a\|^2}$ sample i and output the rank-1 matrix $\|A\|_F^2 \cdot \frac{|A_{i.}\rangle\langle A_{i.}|}{\|\|A_{i.}\|^2\|}$.

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Matrix Chernoff bound – Ahlswede & Winter (2000), Tropp (2010)

Let $B \in R^{n\times n}$ and suppose that $\mathbb{E}[X] = B$, and $\|X - B\| \le \gamma$.
If $X_1, X_2, \ldots$ are iid copies of X, then

$$\mathbb{P}\left(\left\|B - \frac{1}{t}\sum_{i=1}^t X_i\right\| > \varepsilon\right) \le 2n \exp\left(-\frac{\varepsilon^2 t}{3\gamma^2}\right).$$

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(Rejection) sample from the linear combination $x^{(1)} + x^{(1)}$

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Open questions

Better classical algorithms? Better quantum algorithms?

Is there hope for a genuine quantum speedup?

Topological data analysis: Lloyd, Garnerone, and Zanardi (2016),

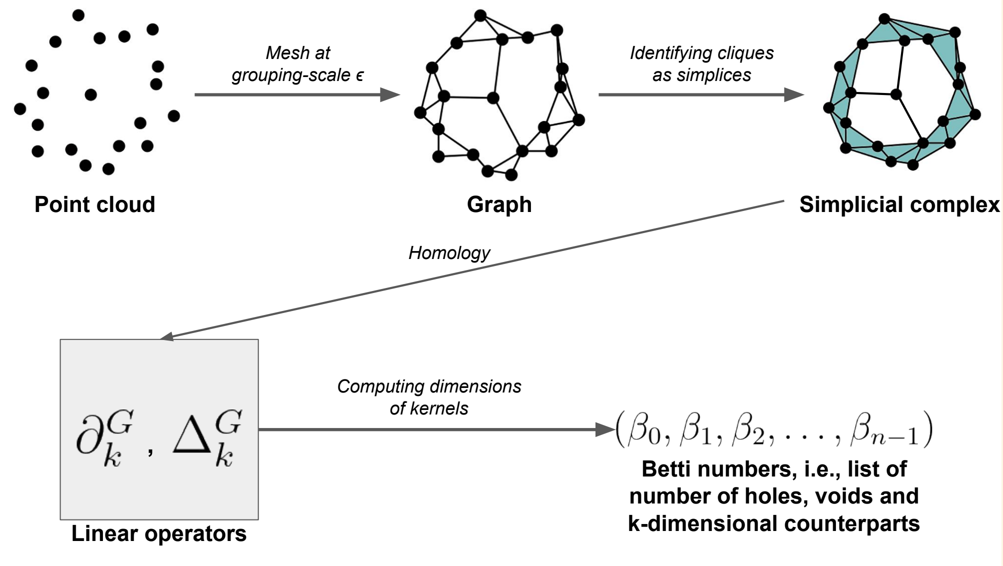

Image from Gyurik, Cade, Dunjko arXiv:2005.02607 (2020)

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Pay-off matrix of Alice is $A \in \mathbb{R}^{m \times n}$. Expected pay-off for strategies x, y: $x^T A y$

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Idea: quantum rejection sampling.

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(Note: Can be improved to $\widetilde{\mathcal{O}}\left((\sqrt{n}+\sqrt{m})/\varepsilon^3\right)$ by using approximate counting.)

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Brandão et al., van Apeldoorn et al. 2016-18 quantum solver $\widetilde{O}\Big((\sqrt{n} + \sqrt{m})(Rr/\varepsilon)^5\Big)$

Learning from quantum data

Quantum principal component analysis (PCA)

Suppose as input we get a copy of a quantum state $|\psi_i\rangle$ with probability p_i.

- The mixed input quantum state is $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
- (For simplicity let us assume $\langle\psi_i, \psi_j\rangle = \delta_{ij}$)
- $\mathcal{O}\left(t^2/\varepsilon\right)$ copies enable implementing ε-approximately $e^{it\rho}$ see "Quantum principal component analysis" by Lloyd, Mohseni, Rebentrost (2013) [Exercise 5: 18.7]
- Using phase estimation we can mark the input states $|\psi_i\rangle|0\rangle \mapsto |\psi_i\rangle|p_i\rangle$

Advantage with quantum memory

Without quantum memory at least $\sim 2^{n/2}$ experiments are needed to learn a fixed property of the principal component of an unknown n-qubit quantum state, while a constant number of experiments suffice when two copies can be jointly processed.

Quantum advantage in learning from experiments: **Huang**, Broughton, Cotler, Chen, Li, Mohseni, Neven, Babbush, Kueng, Preskill, McClean (2021)