A gentle introduction to quantum complexity theory[1]

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[1] https://groups.uni-paderborn.de/fg-qi/courses/UPB_QCOMPLEXITY/2020/UPB_QCOMPLEXITY_syllabus.html for course notes/Youtube videos

Goal

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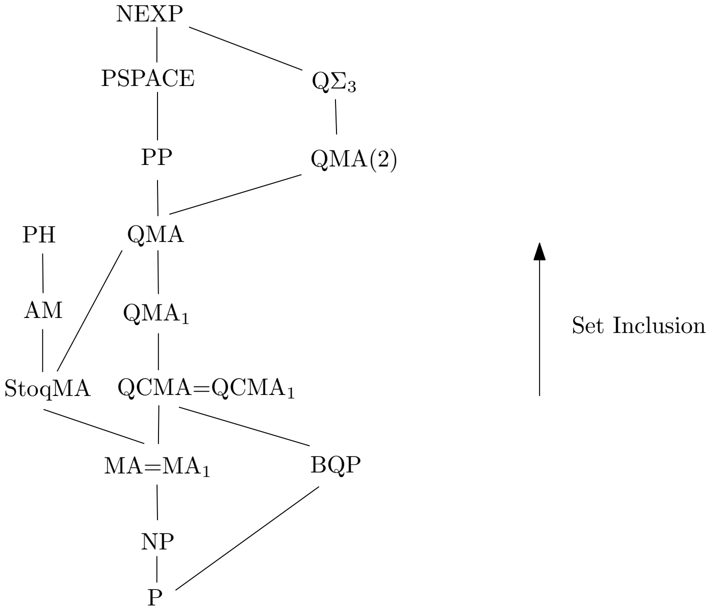
High-level steps:

1. Formal model for computation
2. Complexity theory within this model (classical and quantum)

Preview


NEXP
PSPACE
QΣ3
PP
QMA(2)
PH
QMA
AM
QMA1
StoqMA
QCMA=QCMA1
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BQP
NP
P
Set Inclusion

Outline

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Computational complexity theory

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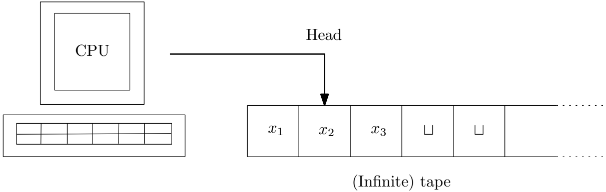
Turing machine (TM)



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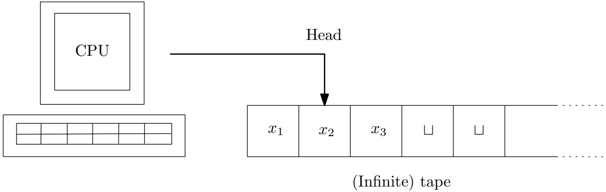

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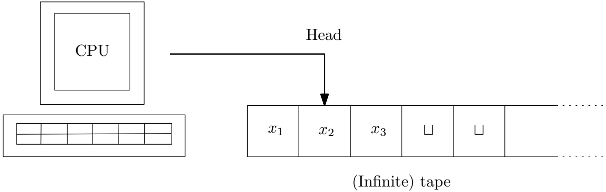

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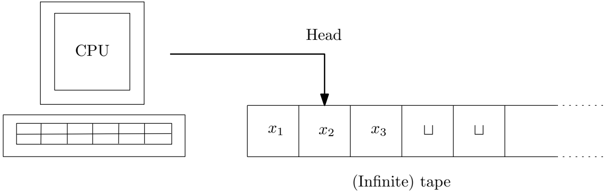

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- What is n? Input size, i.e. $x \in \{0,1\}^n$.

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  - Computation space: Number of tape cells TM uses on tape
- Robust: Power of model unchanged under minor modifications (e.g. 2 tapes instead of 1)
- Church-Turing thesis:

  If there exists a mechanical process for computing function $f : \{0,1\}^* \to \{0,1\}^*$, then there exists a Turing machine computing f.

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- Formally,

$$A_{\text{yes}} = \left\{(x, y, t) \in \mathbb{Z}^3 \mid xy \leq t\right\}, \qquad A_{\text{no}} = \{0,1\}^* \setminus A_{\text{yes}}.$$

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- Aside: Fastest multiplication algorithm is $O(n \log n)$ [Harvey, van der Hoeven, 2019].

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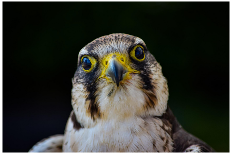
- Is FACTOR $\in$ P?



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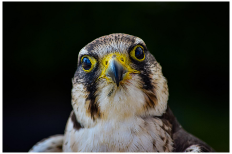

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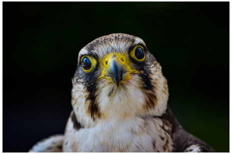

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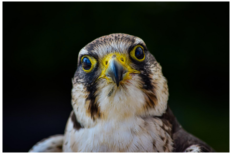

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Bonus: In contrast to FACTOR, checking if x has *any* non-trivial factor is in P [Agrawal, Kayal, Saxena 2002]

Sanity check

Why isn't the naive brute force algorithm poly-time?

Input: $(x, t) \in \mathbb{Z}^2$
Output: Factor $y \in \mathbb{Z}$ of x with $y \leq t$, if one exists

1. Set $k = 2$
2. While $(k < t)$
   a) If $x \mod k = 0$ then return k
   b) $k = k + 1$
3. Return "no factor found"

Runtime: $\approx$ num loop iterations $O(x)$.

Exercise 1: Why is this not poly-time?

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Observe: $\text{P} \subseteq \text{NP}$.

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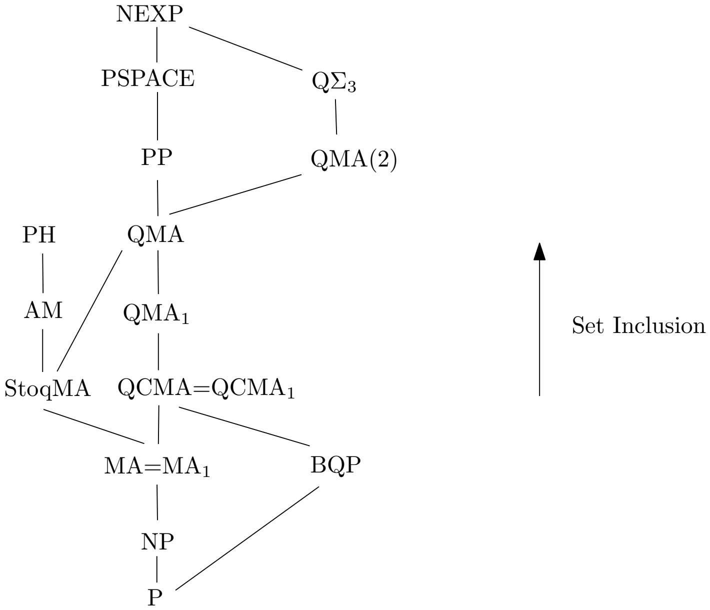
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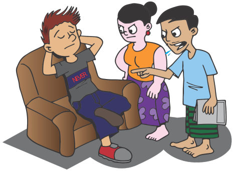
Moral code of complexity theorists: Let someone else solve your problem



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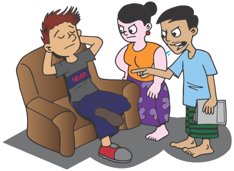

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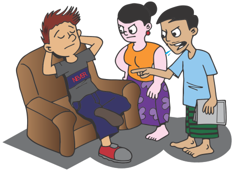

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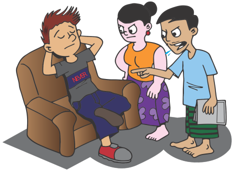

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- Implication: If $A \leq_p B$, then if $B \in \mathrm{P} \Rightarrow A \in \mathrm{P}$.
- Exercise 2: Show that MULTIPLY reduces to ADD $= \left\{(x_1, \ldots, x_k, t) \in \mathbb{Z}^{k+1} \mid k \geq 0 \text{ and } \sum_{i=1}^{k} x_k \leq t\right\}$.
  Is your reduction poly-time?

NP-complete problems

"Strongest/hardest" problems in NP

Formally:

- $B = (B_{\text{yes}}, B_{\text{no}})$ is NP-hard if for all $A = (A_{\text{yes}}, A_{\text{no}}) \in \text{NP}$, $A \leq_p B$.
  - Implication: $B \in \text{P} \Rightarrow \text{P} = \text{NP}$.

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- B is NP-complete if B is NP-hard and $B \in \text{NP}$.
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  - Implication: B "characterizes" the power of NP.
- Cook-Levin Theorem: 3-SAT is NP-complete [Cook 1971, Levin 1973]

3-SAT

Input: Boolean formula $\phi : \{0,1\}^n \rightarrow \{0,1\}$ in "3-Conjunctive Normal Form (3-CNF)", e.g.

$$\phi = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_4 \vee \overline{x_1} \vee \overline{x_9}) \cdots (\overline{x_1} \vee x_5 \vee \overline{x_2})$$

Output: Is there a "satisfying assignment", i.e. $\exists x \in \{0,1\}^n$ such that $\phi(x) = 1$?

Exercise 3: Show that 3-SAT is NP-hard even if each variable x_i appears at most 3 times in ϕ.

Exercise 4: What is the complexity of 3-SAT if each variable x_i appears exactly 3 times in ϕ? (Hint: Google "Hall's marriage theorem".)

Outline

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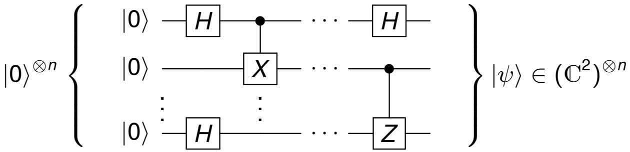
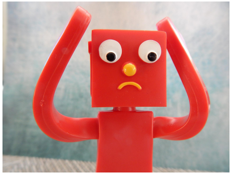
Quantumly: Work with poly(n)-size quantum circuit implementing n-qubit unitaries U, e.g.





What happened to our beloved TMs?

The computational model

What computational model to use for quantum complexity theory?

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    If the poly-size circuit is hard to find, not very useful for solving problems ☹

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  - Solution: Use "poly-time uniformly generated" circuits.

Remark: All quantum circuits in this lecture are sequences of 1- and 2-qubit gates.

P-uniform quantum circuit family

A family of quantum circuits $\{Q_n\}$ is P-uniform if there exists a poly-time TM M, which given as input 1^n, outputs a classical description of Q_n via a sequence of 1- and 2-qubit gates.

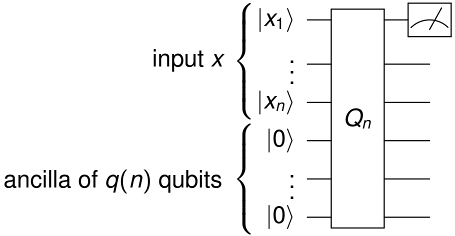
Exercise 6: Why does M get 1^n as input, instead of n written in binary?

Henceforth: Use P-uniform quantum circuit families, not TMs.

Bounded-error quantum polynomial-time (BQP)

Promise problem $\mathbb{A} = (A_{\mathrm{yes}}, A_{\mathrm{no}}, A_{\mathrm{inv}}) \in \mathrm{BQP}$ if $\exists$ P-uniform quantum circuit family $\{Q_n\}$ and polynomial q as below. The first output qubit of Q_n is measured in the standard basis and returned. For any input $x \in \{0,1\}^*$:

- (YES case) If $x \in A_{\mathrm{yes}}$, then Q_n outputs 1 with probability at least $2/3$.
- (NO case) If $x \in A_{\mathrm{no}}$, then Q_n outputs 1 with probability at most $1/3$.
- (Invalid case) If $x \in A_{\mathrm{inv}}$, Q_n outputs 0 or 1 arbitrarily.



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Quantum complexity classes typically promise classes

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  - Intuition: $\text{poly}(n)$ runs of quantum circuit cannot distinguish[2] YES vs NO thresholds like

$$\frac{1}{2}+\frac{1}{2^n} \quad \text{versus} \quad \frac{1}{2}-\frac{1}{2^n}.$$

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  - Exercise 7: Chernoff bound has "exponential scaling". Why does it not suffice above?

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Outline

The Pikachu of BQP

Linear system solving:

- Input: Invertible $A \in \mathbb{C}^{N\times N}$ and target vector $\mathbf{b} \in \mathbb{C}^N$
- Output: $\mathbf{x} \in \mathbb{C}^N$ such that $A\mathbf{x} = \mathbf{b}$.

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- A and $\mathbf{x}$ given explicitly in matrix form $\Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$ classically in time $\text{poly}(N)$
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  - Exercise 8: Is linear system solving thus in P?
- A represented "succinctly" via "query-access" and $\mathbf{b}$ given via quantum circuit?

Matrix inversion problem (MI)

Input:

- $O(1)$-sparse row-computable invertible Hermitian matrix[3] $A \in \mathbb{C}^{N\times N}$
- A specified via polylog$N$-time TM M which, given row index $r \in [N]$ of A, outputs entries of row r of A

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Proof steps:

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Hamiltonian simulation [Low, Chuang 2017]

Given d-sparse H, simulation time $t \geq 0$, and $\epsilon > 0$, can simulate e^{iHt} up to error ϵ and success probability at least $1 - 2\epsilon$ in time[a]

$$O\left(td\,\|H\|_{\max} + \frac{\log(1/\epsilon)}{\log\log(1/\epsilon)}\right).$$

[a] Query complexity. Gate complexity has $O(n)$ overhead.

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Given precision k, and ability to efficiently compute controlled-U^{2^K} for $1 \leq K \leq k$, can map

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$$|0^k\rangle|\psi_j\rangle \mapsto |\widetilde{\lambda}_j\rangle|\psi_j\rangle \quad \Rightarrow \quad |0^k\rangle \sum_j \alpha_j |\psi_j\rangle \mapsto \sum_j \alpha_j |\widetilde{\lambda}_j\rangle|\psi_j\rangle,$$

where $\widetilde{\lambda}_j$ is λ_j up to k bits.

Exercise 9a: Given n-qubit unitary U, can we efficiently compute U^{2^n} in general?

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- Conditioned on the first register, rotate a new single-qubit ancilla as follows:

$$\sum_{j=1}^{N} \alpha_j |\lambda_j\rangle |\psi_j\rangle \left(\sqrt{1 - \frac{1}{\lambda_j^2 \kappa^2(A)}} |0\rangle + \left(\frac{1}{\lambda_j \kappa(A)} \right) |1\rangle \right) \in (\mathbb{C}^2)^{\otimes n} \otimes \mathbb{C}^N \otimes \mathbb{C}^2.$$

Exercise 9b: Google “condition number”, learn about it.

Exercise 10: Assume $\|A\|_\infty = 1$. Show $1/\kappa(A) \leq 1/(\lambda_j \kappa(A)) \leq 1$. Thus, amplitudes above well-defined.

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Step 3: Eigenvalue reinsertion

- Measure third register in standard basis, postselect on outcome 1, discard third register:

$$\sum_{j=1}^{N} \alpha_j \left(\frac{1}{\lambda_j} \right) |\psi_j\rangle \propto A^{-1}|b\rangle \in \mathbb{C}^N.$$

Exercise 11. Prove that probability of obtaining outcome 1 is at least $1/\kappa^2(A)$.

Exercise 12. What is the expected number of repetitions for postselection to succeed? Can we improve this with amplitude amplification?

Runtime

If we run QPE to get additive inverse poly error for phases, runtime is

$$\widetilde{O}(\kappa(A)(T_b + s^2 \log^2(N)))$$

for T_b the number of gates to prepare $|b\rangle$, N the dimension of A, and $\log N$ the number of qubits.

Implication:

- When $\kappa(A), T_b, s \in \text{polylog}(N)$, exponentially faster than classically solving the entire $N \times N$ system.
- For definition of MI, suffices to obtain MI $\in \mathrm{BQP}$.

Exercise 13**. Although the quantum algorithm can give exponential speedups, why is it incorrect to directly compare it to classical linear system solvers?

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MI is BQP-hard

Goal: Show that any BQP computation V poly-time reducible to an instance A of MI.

Starting point: Let $V = V_m \cdots V_1$ be a BQP circuit on n qubits, $N = 2^n$. Assume WLOG m is power of 2.

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- Recall Maclaurin series $\frac{1}{1-x} = \sum_{l=0}^{\infty} x^l$ for $|x| < 1$.
- We could apply this to any normal matrix U with $\|U\|_\infty < 1$ to get

$$(I - U)^{-1} = \sum_{l=0}^{\infty} U^l.$$

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- Define:

$$U = \sum_{t=0}^{m-1} |t+1\rangle\langle t| \otimes V_{t+1} + \sum_{t=m}^{2m-1} |t+1 \bmod 2m\rangle\langle t| \otimes V^{\dagger}_{2m-t} \in \mathcal{U}((\mathbb{C}^2)^{\otimes \log m} \otimes (\mathbb{C}^2)^{\otimes n}),$$

Exercise 12: Check that U is unitary.

Exercise 13: Check that $U^m|0^{\log m}\rangle|0^n\rangle = |m\rangle V|0^n\rangle$.

Implication: Measuring first qubit of second register of $U^m|0^{\log m}\rangle|0^n\rangle$ simulates measuring output qubit of V!

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Exercise 15: I cheated less slightly somewhere else on this slide. Where did I make a bigger boo boo?

Final exercises for MI

Construction *almost* works, but for 3 issues to check:

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4. I cheated again. There is a 4th issue — A must be Hermitian. But I will spare you these details.

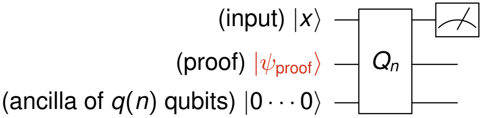
Outline

QMA

Quantum Merlin-Arthur (QMA)

Promise problem $\mathbb{A} = (A_{\text{yes}}, A_{\text{no}}, A_{\text{inv}}) \in$ QMA if $\exists$ P-uniform quantum circuit family $\{Q_n\}$ and polynomials p,q:

- (YES case) If $x \in A_{\text{yes}}$, $\exists$ proof $|\psi_{\text{proof}}\rangle \in (\mathbb{C}^2)^{\otimes p(n)}$, such that Q_n accepts with probability at least $2/3$.
- (NO case) If $x \in A_{\text{no}}$, then $\forall$ proofs $|\psi_{\text{proof}}\rangle \in (\mathbb{C}^2)^{\otimes p(n)}$, Q_n accepts with probability at most $1/3$.
- (Invalid case) If $x \in A_{\text{inv}}$, Q_n may accept or reject arbitrarily.



Error reduction for QMA

Weak error reduction (the "obvious" type)

- Idea: Given $\text{poly}(n)$ copies of $|\psi_{\text{proof}}\rangle$, repeat verification $\text{poly}(n)$ times and take majority vote

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Obstacle: No-cloning theorem says we cannot *copy* $|\psi_{\text{proof}}\rangle$...

Marriot-Watrous strong error reduction

- Set $i = 0$.
- Do while $i \leq N$:
  - (Run verification Q_n) Run Q_n and measure output qubit to obtain bit y_i. Set $i = i + 1$.
  - (Run Q_n in reverse) Run $Q_n^\dagger$ and measure whether input "resets" to x and ancillae to $|0 \cdots 0\rangle$. If yes, set $y_i = 1$, else set $y_i = 0$. Set $i = i + 1$.
- (Postprocessing) If the number of indices $i \in \{0, \ldots, N-1\}$ such that $y_i = y_{i+1}$ is at least $N/2$, accept. Otherwise, reject.

Outline

Remember this?

(Time-independent) Schrödinger equation

Time evolution of any n-qubit system governed by Hermitian matrix $H \in \mathcal{L}(\mathbb{C}^2)^{\otimes n}$, called a Hamiltonian:

$$i\frac{d|\psi\rangle}{dt} = H|\psi\rangle \quad \xrightarrow{\text{solve}} \quad |\psi_t\rangle = e^{-iHt}|\psi_0\rangle \quad (\leftarrow \text{ unitary!})$$

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Question: What kind of Hamiltonians H appear in nature?

k-local Hamiltonian

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An n-qubit Hermitian operator $H = \sum_i H_i \in \mathcal{L}((\mathbb{C}^2)^{\otimes n})$, where

- each H_i is a $2^k \times 2^k$ matrix for $k \in O(1)$, i.e. a quantum constraint,

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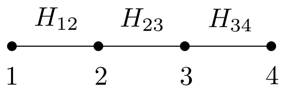
Example. Let $H_{ij} = X_i \otimes X_j + Y_i \otimes Y_j + Z_i \otimes Z_j \in \mathcal{L}(\mathbb{C}^4)$.



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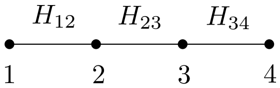

Then, $H = H_{12} \otimes I_{34} + I_1 \otimes H_{23} \otimes I_4 + I_{12} \otimes H_{34} \in \mathcal{L}(\mathbb{C}^{16})$.

Quantum constraint satisfaction

k-local Hamiltonian problem (k-LH)

- Input: k-local Hamiltonian H on n qubits, thresholds $0 \leq \alpha \leq \beta$ s.t. $|\alpha - \beta| \geq 1/\operatorname{poly}(n)$
- Promise: $\lambda_{\min}(H) \leq \alpha$ or $\lambda_{\min}(H) \geq \beta$
- Output: Decide whether $\lambda_{\min}(H) \leq \alpha$ or $\lambda_{\min}(H) \geq \beta$

- Canonical QMA-complete problem!
- Motivation: Show superfluid helium video

https://www.youtube.com/watch?v=2Z6UJbwxBZI

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Variants:

- PSPACE-complete for $|\alpha - \beta| \geq 1/\exp(n)$ [Fefferman, Lin, 2016]

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- PSPACE-complete for $|\alpha - \beta| \geq 1/\exp(n)$ [Fefferman, Lin, 2016]
- QMA-hard for $|\alpha - \beta| \in \Omega(1)$?

Quantum PCP conjecture! (see [Aharonov, Arad, Vidick, 2013] for survey)

Kitaev's quantum Cook-Levin theorem

Goal: Map U to instance $(H, \alpha, \beta, |\psi\rangle)$ of LH such that $\beta - \alpha \geq 1/\operatorname{poly}(n)$ and

$$\begin{aligned} \text{if } U \text{ accepts } x \quad &\Longrightarrow \quad \lambda_{\min}(H) \leq \alpha \\ \text{if } U \text{ rejects } x \quad &\Longrightarrow \quad \lambda_{\min}(H) \geq \beta \end{aligned}$$

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- Let $U = U_m \cdots U_1$ be a QMA circuit verifying proof $|\psi_{\text{proof}}\rangle$.
- Design local terms H_i to force ground state to be history state:

$$|\psi_{\text{hist}}\rangle = \frac{1}{\sqrt{m+1}} \sum_{t=0}^{m} U_t \cdots U_1 |\psi_{\text{proof}}\rangle_A |0 \cdots 0\rangle_B |t\rangle_C$$

A: proof register B: ancilla register C: clock register

Feynman-Kitaev circuit-to-Hamiltonian construction

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Question: How to check time propagation, i.e. U_t applied at time t?

The propagation term

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Correctness

Completeness: By design,

$$\langle\psi_{\text{hist}}|H_{\text{in}} + H_{\text{prop}} + H_{\text{out}} + H_{\text{stab}}|\psi_{\text{hist}}\rangle \sim 0 + 0 + 0 + \frac{1 - \Pr(U \text{ accepts } x)}{\text{poly}(m)} \sim \text{“small”}.$$

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Soundness:

- Goal: Show $\lambda_{\min}(H_{\text{in}} + H_{\text{prop}} + H_{\text{out}} + H_{\text{stab}}) \geq$ “large”.
- Problem: $H_{\text{in}} + H_{\text{out}}$ and H_{prop} do not commute (i.e. cannot add $\lambda_{\min}(H_{\text{in}} + H_{\text{out}})$ and $\lambda_{\min}(H_{\text{prop}})$)!

Geometric Lemma

Let $A_1, A_2 \succeq 0$, and let v lower bound the minimum non-zero eigenvalues of both A_1 and A_2. Then,

$$\lambda_{\min}(A_1 + A_2) \geq 2v \sin^2 \frac{\angle(\mathrm{Null}(A_1), \mathrm{Null}(A_2))}{2},$$

where the angle between spaces $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$\angle(\mathcal{X}, \mathcal{Y}) := \arccos\left(\max_{\substack{|x\rangle \in \mathcal{X}, |y\rangle \in \mathcal{Y} \\ \| |x\rangle \|_2 = \| |y\rangle \|_2 = 1}} |\langle x|y\rangle| \right).$$

Recall:

- $\mathrm{Null}(H_{\mathrm{in}} + H_{\mathrm{out}})$ - correct initialization and correct input
- $\mathrm{Null}(H_{\mathrm{prop}})$ - correct time propagation

Outline

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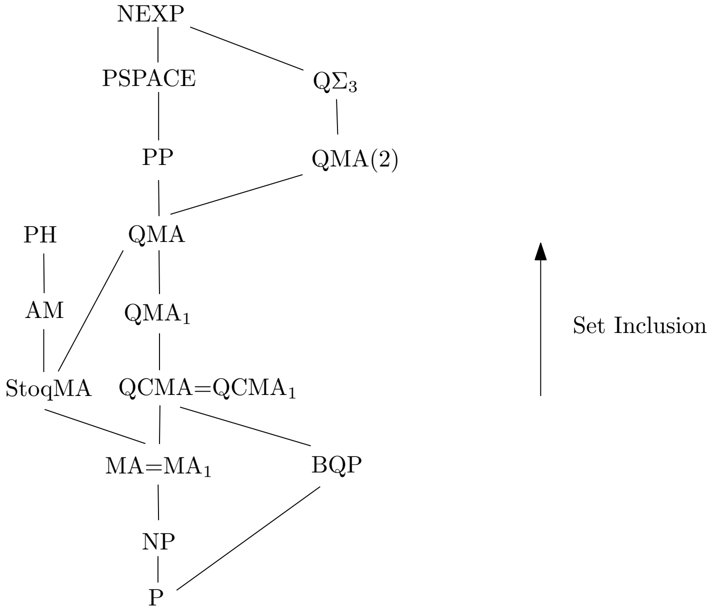
- (Dopey) StoqMA: QMA with $\{|0\rangle, |+\rangle\}$ ancillae, classical gates, measurement in X basis

Relationships



Grumpy

QMA(2)

What does an "unentangled" proof $|\psi_1\rangle \otimes |\psi_2\rangle$ buy us?

QMA(2)

Promise problem $\mathbb{A} = (A_{\text{yes}}, A_{\text{no}}, A_{\text{inv}}) \in \text{QMA}(2)$ if there exists P-uniform quantum circuit family $\{Q_n\}$ s.t.:

- (YES) If $x \in A_{\text{yes}}$, $\exists$ proof $|\psi_1\rangle \otimes |\psi_2\rangle \in (\mathbb{C}^2)^{\otimes \text{poly}(n)} \otimes (\mathbb{C}^2)^{\otimes \text{poly}(n)}$, s.t. Q_n accepts w.p. $\geq 2/3$.
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It's 2022. What's the holdup?

Apples to apples

For both classes:

$$\Pr(Q_n \text{ accepts } |\psi\rangle) = \mathrm{Tr}\left(|1\rangle\langle 1|_{A_1} \otimes I_B(Q_n|\psi\rangle_A|0\cdots 0\rangle_B)(\langle\psi|_A\langle 0\cdots 0|_B Q_n^\dagger)\right)$$

[5] For general Hermitian matrices M, not necessarily M_{acc} arising from some Q_n [Gurvits 2003]

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Complexity	poly-time in dimension of M_{acc}	NP-complete[5] dimension of M_{acc}

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Open question: Why does “unentanglement” help compress proof lengths?

Relationship to Quantum NPSPACE

- Classically: $\text{PSPACE} = \text{NPSPACE}$ [Savitch, 1970]
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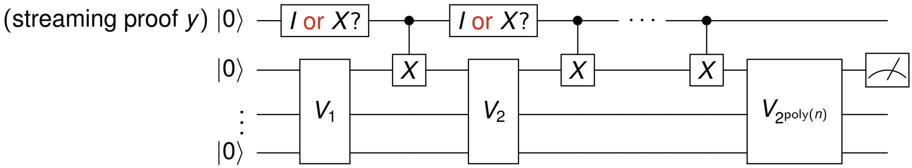
    - ★ Problem: NPSPACE requires exponential length proof!

Question: How to define "Quantum NPSPACE" with exp-length proof?

Streaming QCMASPACE (SQCMASPACE)

Promise problem $A = (A_{\text{yes}}, A_{\text{no}}) \in \text{SQCMASPACE}$ if there exists a poly-time succinctly generated quantum circuit family $\{Q_n\}$, thresholds α, β satisfying $\alpha - \beta \geq 2^{-poly(n)}$ s.t.:

- (YES case) If $x \in A_{\text{yes}}$, $\exists$ classical streaming proof $y \in \{0,1\}^{2^{\text{poly}(n)}}$, s.t. Q_n accepts with probability $\geq \alpha$.
- (NO case) If $x \in A_{\text{no}}$, $\forall$ classical streaming proofs $y \in \{0,1\}^{2^{\text{poly}(n)}}$, Q_n accepts with probability $\leq \beta$.



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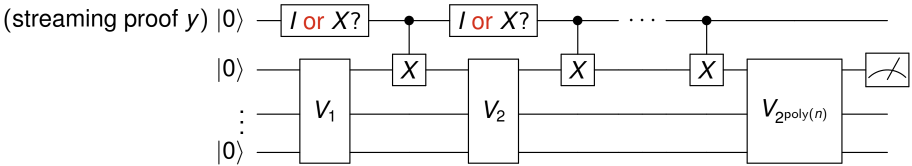

- $\text{SQCMASPACE} = \text{NEXP}$, even with 1 vs $1/2$ promise gap [G, Rudolph, 2022]
- Question: Embed exp-length streaming proofs into poly-size history state construction?

Recall: Circuit-to-Hamiltonian construction for QMA

$$|\psi_{\text{hist}}\rangle = \frac{1}{\sqrt{m+1}} \sum_{t=0}^{m} U_t \cdots U_1 |\psi_{\text{proof}}\rangle_A |0\cdots 0\rangle_B |t\rangle_C$$

Define $H = H_{\text{in}} + H_{\text{out}} + H_{\text{prop}} + H_{\text{stab}}$ such that

H_{in}:	Correct ancilla initialization at time $t=0$	$\Rightarrow$	$\langle\psi_{\text{hist}}\|H_{\text{in}}\|\psi_{\text{hist}}\rangle = 0$
H_{prop}:	Gate U_t applied at time t	$\Rightarrow$	$\langle\psi_{\text{hist}}\|H_{\text{prop}}\|\psi_{\text{hist}}\rangle = 0$
H_{stab}:	Clock register C encoded correctly in unary	$\Rightarrow$	$\langle\psi_{\text{hist}}\|H_{\text{out}}\|\psi_{\text{hist}}\rangle = 0$
H_{out}:	Penalize rejecting computation U at time $t=m$	$\Rightarrow$	$\langle\psi_{\text{hist}}\|H_{\text{out}}\|\psi_{\text{hist}}\rangle \sim \frac{1-\Pr(U \text{ accepts } x)}{\text{poly}(m)}$

Define for each $t \in \{0,\ldots,m-1\}$:

$$H^{U_t}_{\text{prop},t} = -U_t \otimes |t\rangle\langle t-1|_C - U_t^\dagger \otimes |t-1\rangle\langle t|_C + I \otimes |t-1\rangle\langle t-1|_C + I \otimes |t\rangle\langle t|_C,$$

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Problem: Need to know each gate U_t in advance. But "proof gates" *a priori* unknown.

Using history states to encode the future

$$H^{U} := -U \otimes |t\rangle\langle t-1|_C - U^{\dagger} \otimes |t-1\rangle\langle t|_C + I \otimes |t-1\rangle\langle t-1|_C + I \otimes |t\rangle\langle t|_C.$$

Idea [G, Rudolph, 2022]: Use “unentanglement”, i.e. try to force prover to send $|\psi_{\text{hist}}\rangle \otimes |\psi_{\text{hist}}\rangle$.

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Thought experiment: Imagine parallel universes L and R, s.t. L streams 0, R streams 1.

round	L	R
1	0	
2	0	
3		1
4	0	
5		1

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Unentangled constraint to simulate this: $H_L^I \otimes H_R^X$.

Why?

$$(H_L^I \otimes H_R^X)|\psi\rangle_L \otimes |\phi\rangle_R = 0 \quad \Leftrightarrow \quad H_L^I|\psi\rangle = 0 \text{ OR } H_R^X|\phi\rangle = 0.$$

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$$\left(H_L^I \otimes H_R^X + H_L^X \otimes H_R^I\right)|\psi\rangle_L \otimes |\phi\rangle_R = 0 \quad \Leftrightarrow \quad \begin{array}{l}(H_L^I|\psi\rangle = 0 \text{ AND } H_R^I|\phi\rangle = 0) \text{ OR} \\ (H_L^X|\psi\rangle = 0 \text{ AND } H_R^X|\phi\rangle = 0)\end{array}$$

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In words:

- Each universe can stream either proof bit, as long as both universes choose the same bit!

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In words:

- Each universe can stream either proof bit, as long as both universes choose the same bit!
- Exploited quadratic property of unentanglement to simulate logical EQUALS function on L vs R.
- Gives intuitive explanation as to why unentanglement helps!

Full construction

$$\widetilde{H} = \Delta_{\text{in}}\widetilde{H}_{\text{in}} + \Delta_{\text{prop}}\widetilde{H}_{\text{prop}} + \Delta_{\text{sym}}\widetilde{H}_{\text{sym}} + \widetilde{H}_{\text{out}} \tag{1}$$

$$\widetilde{H}_{\text{in}} = (H_{\text{in}})_L \otimes I_R + I_L \otimes (H_{\text{in}})_R \tag{2}$$

$$\widetilde{H}_{\text{prop}} = \sum_{t=1}^{m} \widetilde{H}_t, \quad \text{where } \widetilde{H}_t \text{ is defined as} \tag{3}$$

$$\widetilde{H}_t = \begin{cases} (H_t^l)_L \otimes (H_t^{iX})_R + (H_t^{iX})_L \otimes (H_t^l)_R & \text{if } t \in P \\ (H_t)_L \otimes I_R + I_L \otimes (H_t)_R & \text{if } t \notin P \end{cases} \tag{4}$$

$$\widetilde{H}_{\text{out}} = (H_{\text{out}})_L \otimes I_R + I_L \otimes (H_{\text{out}})_R \tag{5}$$

$$\widetilde{H}_{\text{sym}} = I - P_{LR}^{\text{sym}} \quad \text{for} \quad P_{LR}^{\text{sym}} = \frac{1}{2}\left(I_{LR} + \sum_{xy} |xy\rangle\langle yx|_{LR}\right), \tag{6}$$

- Recall: Analysis not an eigenvalue analysis!

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- Recall: Analysis not an eigenvalue analysis!
- With more work: Can encode any multi-prover interactive proof into QMA(2), but promise gap scales $1/\exp$ with communication length
- Upshot: First systematic "compression" of long proofs into small history states, but does not yet resolve $\text{QMA}(2)$ versus NEXP (our construction requires $1/\exp$ gap for $\text{QMA}(2)$ to capture NEXP).

Summary

- Turing machines rule theoretical computer science
- Quantumly, we use uniformly generated circuit families
- Matrix Inversion is BQP-complete
- Local Hamiltonian problem is QMA-complete
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Thank you and happy quantuming!