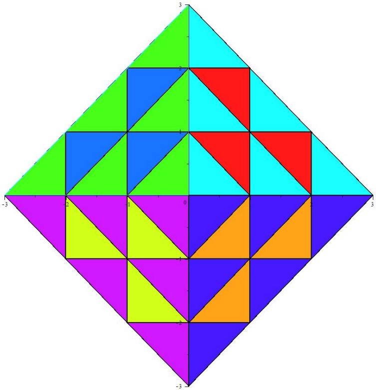
Michèle Vergne : Multiplicities, Quivers, Polytopes

Bochum : 8th Workshop on Algebraic Complexity Theory (April 2025)



Multiplicities, Quivers, Polytopes

The aim of my talk is to describe some general results on multiplicity functions,
and to extend some results (*Knutson-Tao,..., Chindris-Collins-Kline*) on interpreting a multiplicity as the number of points in a polytope to the case of quivers.
This is based on our note
MV, Michael Walter *Moment cone membership for quivers in strongly polynomial time ; arXiv :2303.14821* and previous articles.

I will also discuss more specifically questions on semi-invariants.

Volume of polytopes and number of integral points

Formulae $A = B$ relating volume of polytopes and number of integral points with **M.Brion** *Residue formulae, vector partition functions and lattice points in rational polytopes, 1997*
and with
A. Szenes *Residue formulae for vector partitions and EulerMacLaurin sum, 2003*
BUT ... impossible to compute either A or B.

Less ambitious goals :
Say something on what type of functions are multiplicity functions and their support.
Deciding if a Clebsh-Gordan coefficient $c_{\lambda,\mu,\nu}$ is non zero ?
Deciding if a Kronecker coefficient $g(\lambda, \mu, \nu)$ is non zero ?

$G = (S^1)^r$ acting on a complex vector space H.

We write an element z in $H = \mathbb{C}^N$: $z = \sum_{i=1}^N z_i e^i$.
We write an element g in $G = (S^1)^r$: $g = (e^{i\theta_1}, \dots, e^{i\theta_r})$.

A polynomial function on H is a polynomial function of $z_1, z_2, \dots, z_N$.
Thus $g \in G$ acts on $\mathcal{P} = \mathbb{C}[z_1, z_2, \dots, z_N]$

Let

$$\lambda = (n_1, n_2, \dots, n_r) \in \mathbb{Z}^r, \qquad g^\lambda = e^{in_1\theta_1} \cdots e^{in_r\theta_r}.$$

$$\mathcal{P}_\lambda = \{P \in \mathcal{P}, P(gz) = g^\lambda P(z); \text{for all}\, g \in G, z \in H\}$$

the space of semi-invariant polynomials of weight λ;
Definition

$$m_H(\lambda) := \dim \mathcal{P}_\lambda$$

the multiplicity of the weight λ in $\mathcal{P}$.

Questions

• **A :** Can we give a description of the set Σ_H consisting of the elements λ such that $m_H(\lambda) > 0$, or of the cone generated by Σ_H ?
• **B :** If $\lambda \in \Sigma_H$, can we give a geometric meaning to $m_H(\lambda)$.
• **C :** What type of function is $\lambda \to m_H(\lambda)$ as a function of λ, and eventually of H.

Example 1 : Action of $(S^1)^2$ on $\mathbb{C}^2$; $(z_1, z_2) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$

Consider

$$G = \{g = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}\}$$

The function $p(z_1, z_2) = z_1^{n_1} z_2^{n_2}$ is in $\mathcal{P}_{(n_1, n_2)}$

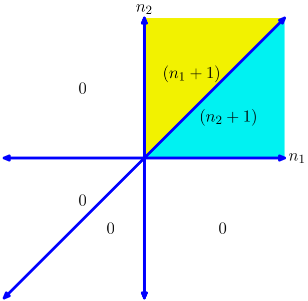
$$Q_2 = \{(x_1, x_2), x_1 \geq 0, x_2 \geq 0\}$$

$$m_H(\lambda) = 1, \text{ if } \lambda \in Q_2 \cap \mathbb{Z}_2.$$

$$m_H(\lambda) = 0 \text{ otherwise}$$

$H = \mathbb{C}^3$, $g(z_1, z_2, z_3) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_1 + i\theta_2} z_3)$

Drawing of the multiplicity function $m_H(n_1, n_2)$. For example $m_H(1, 1) = 2$, since $z_1 z_2$ and z_3 are semi-invariants of weight $(1, 1)$



Remark : the function (defined only on $\mathbb{Z}^2$) is piecewise polynomial and "continuous". It is the restriction of two linear functions $y + 1$ or $x + 1$ on the two cones above, and these functions match on the intersection.

Results : G a torus acting on H

$\mathcal{P}$ the space of polynomial functions on H (or more generally the space of polynomials on a G-invariant affine subvariety M of H)

A : The set Σ_H of elements λ such that $m_H(\lambda) > 0$ generates a rational polyhedral cone.

B : The function $\lambda \to m_H(\lambda)$ is piecewise (quasi) -polynomial on cones C_i and "continuous". The polynomials obtained satisfies a system of difference equations (Dahmen-Micchelli polynomials) with finite number of solutions (De Concini, Procesi,V.)

C : It can be "computed" using Jeffrey-Kirwan residues or using Paradan wall crossing formulae. (Christandl, Doran, Walter,),or by Latte program (de Loera et al).

D : When H is a vector space, the number $m_H(\lambda)$ is the number of integral points in a convex rational polytope, equivalently dimension of a space of holomorphic sections of a line bundle on a toric variety. For M affine subvariety, interpretation in terms of the GIT quotient of M at λ.

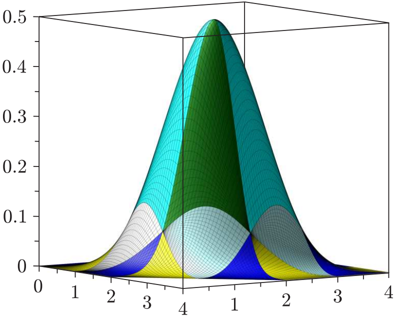
Example : A multiplicity function for the action of $G = S_1^2$ on an affine variety

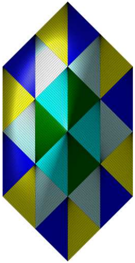
We look at the space $M = \{(X_1, X_2), X_1, X_2 \in End(\mathbb{C}^3), X_1^2 = 0, X_2^2 = 0\}$ and the space $\mathcal{P}$ of polynomial functions on M.
Action of the torus

$$G = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by simultaneous conjugacy (gX_1g^{-1}, gX_2g^{-1}) on the 3 times 3 matrices X_1, X_2.
In the following drawing, we draw the multiplicity function $m_H(n_1, n_2, t)$. of polynomial functions homogeneous of degree t, t in X_1, X_2 and satisfying $P(gX_1g^{-1}, gX_2g^{-1}) = e^{in_1\theta_1}e^{in_2\theta_2}P(X_1, X_2)$

Here is a picture of multiplicities on a slice





Multiplicities for a compact group action

Let G be compact connected Lie group (products of groups $U(n)$ for this talk) acting on a Hermitian vector space H.
Let $\mathcal{P}$ be the space of polynomial functions on H (or on a G-invariant affine subvariety M of H)
Write the decomposition of $\mathcal{P}$ in irreducible representations :

$$\mathcal{P} = \oplus_{\lambda \in \hat{G}} m_H(\lambda) V_\lambda$$

Then $m_H(\lambda)$ is called the multiplicity of the representation V_λ in $\mathcal{P}$.
Determining $m_H(\lambda)$ is a part of invariant theory. Here we assume that the space of G invariant polynomials on H is reduced to the constants. Otherwise $m_H(\lambda)$ is ∞.
An important case is the case where $G = U(n)$, and $\lambda(g) = \det(g)^\sigma$.
Then $m_H(\lambda)$ is the dimension of the space of semi-invariant polynomials $P(gv) = \det(g)^\sigma P(v)$.

Some Results

We assume that there are no invariant polynomials except constants
A : the set $Cone_H$ generated by the elements λ such that $m_H(\lambda) > 0$ is a convex rational polytope.
Guillemin-Sternberg, Mumford
Usually difficult to describe the inequalities defining this cone

B : In general $m_H(\lambda)$ can be interpreted geometrically as a space of holomorphic sections on a *GIT* quotient $H//_\lambda G$, but it not the number of integral points in a polytope. *Plethysm and lattice point counting : T Kahle, M Michalek (2015)*

C : The function $\lambda \mapsto m_H(\lambda)$ is piecewise quasi-polynomial and "continuous" on $Cone_H$.
Singular reduction and quantization E Meinrenken, R Sjamaar , 1999.

An emblematic example : Clebsch-Gordan coefficients

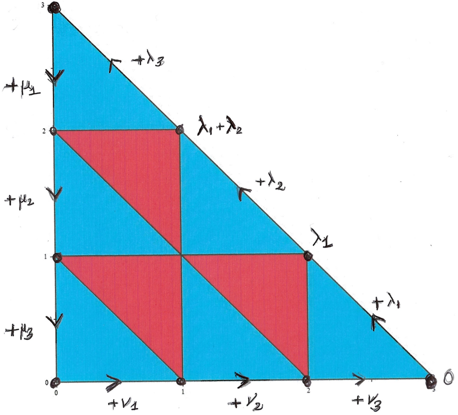
$G = U(n)$, $\lambda = (\lambda(1) \geq \lambda(2) \geq \cdots \geq \lambda(n))$. We write $\lambda \geq 0$ if $\lambda(n) \geq 0$.

Define $|\lambda| = \sum_{i=1}^{n} \lambda(i)$.
Let V_λ be the irreducible representation of G with highest weight λ.
The dual representation has weight
$\lambda^* = (-\lambda(n) \geq -\lambda(n-1) \geq \cdots \geq -\lambda(1))$. We identify holomorphic representations of $Gl(n, \mathbb{C})$ or representations of $U(n)$.

We denote by $C_{\lambda,\mu,\nu}$ the multiplicity of the trivial representation of G in $V_\lambda \otimes V_\mu \otimes V_\nu$.

$C_{\lambda,\mu,\nu} := c_{\lambda,\mu}^{\nu^*}$.
Need $|\lambda| + |\mu| + |\nu| = 0$ *for* $C_{\lambda,\mu,\nu} > 0$
Knutson-Tao described $C_{\lambda,\mu,\nu}$ as the number of integral points in a polytope.

Knutson-Tao Polytope $KT(\lambda, \mu, \nu)$



Knutson-Tao Polytope inequalities

Variables $(t_{i,j}) \in \mathbb{R}^{(n+1)(n+2)/2}$. Example $n = 3$

Boundary equations

$$[t_{1,2} - t_{0,3} = \lambda_1, t_{2,1} - t_{1,2} = \lambda_2, t_{3,0} - t_{2,1} = \lambda_3]$$

$$[t_{2,0} - t_{3,0} = \mu_1, t_{1,0} - t_{2,0} = \mu_2, t_{0,0} - t_{1,0} = \mu_3]$$

$$[t_{0,1} - t_{0,0} = \nu_1, t_{0,2} - t_{0,1} = \nu_2, t_{0,3} - t_{0,2} = \nu_3]$$

Weyl inequalities $\lambda_1 \geq \lambda_2 \geq \lambda_3$,$\mu_1 \geq \mu_2 \geq \mu_3$, $\nu_1 \geq \nu_2 \geq \nu_3$

Rhombi inequalities ≤ 0

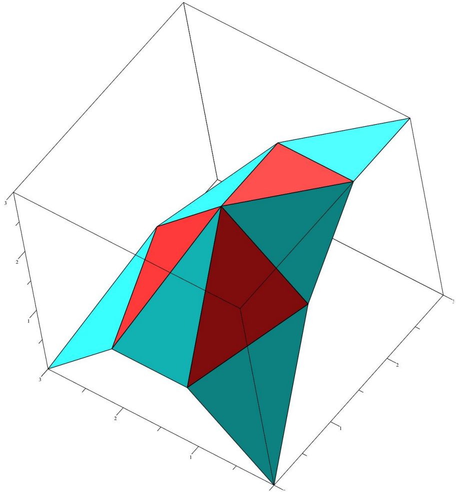
$$[t_{1,1} - t_{1,0} + t_{0,0} - t_{0,1}, t_{1,2} - t_{1,1} + t_{0,1} - t_{0,2}, t_{2,1} - t_{2,0} + t_{1,0} - t_{1,1},$$

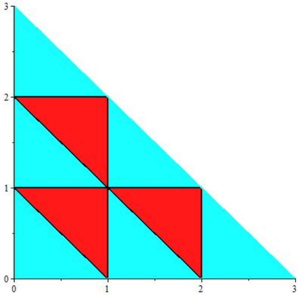
$$t_{1,0} - t_{1,1} + t_{0,2} - t_{0,1}, t_{1,1} - t_{1,2} + t_{0,3} - t_{0,2}, t_{2,0} - t_{2,1} + t_{1,2} - t_{1,1},$$

$$t_{0,1} - t_{1,1} + t_{2,0} - t_{1,0}, t_{0,2} - t_{1,2} + t_{2,1} - t_{1,1}, t_{1,1} - t_{2,1} + t_{3,0} - t_{2,0}]$$

Meaning of inequalities

$t_{i,j}$ height function for a roof made with triangle tiles :





Results of Knutson-Tao for $G = U(n)$

A : The set $Cone_n = \{(\lambda, \mu, \nu)\}$ with $C_{\lambda,\mu,\nu} > 0$ is described by the "explicit" inequalities conjectured by Horn (see later).
But the Horn conditions are determined by recurrence, and exponential number of them, so no way to decide in polynomial time if $C_{\lambda,\mu,\nu} > 0$ *using Horn inequalities*.

B : if $C_{N\lambda,N\mu,N\nu} > 0$ for some $N \geq 1$, then $C_{\lambda,\mu,\nu} > 0$.
Saturation property

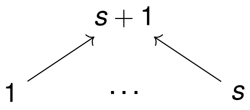
C : $C_{\lambda,\mu,\nu}$ is the number of integral points in the convex polytope $KT(\lambda, \mu, \nu)$ and $C_{\lambda,\mu,\nu} > 0$ if and only the polytope $KT(\lambda, \mu, \nu)$ is non empty.

D : The polytope $KT(\lambda, \mu, \nu)$ is described by linear inequalities with coefficients 0 or ± 1. Consequently, there is a strongly polynomial algorithm to determine if $C_{\lambda,\mu,\nu} > 0$
Mulmuley-Narayanan-Sohoni, ...

Representations of quivers

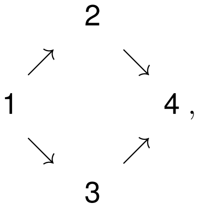
Quiver Q : Q_0 set of vertices, Q_1 set of arrows. Denote by $x \in Q_0$ vertices, $a \in Q_1$ arrows

The *Horn quiver* $Horn_s$:



(1)

The quiver Q_4 :



(2)

Why quivers : easier and more general

Representation of Q_4 of dimension $\mathbf{n} = (n_1, n_2, n_3, n_4)$

By definition a representation of $Q = Q_4$ with dimension vector $\mathbf{n} = (n_1, n_2, n_3, n_4)$ is the following data :

$$\begin{array}{ccccc} & & 2 & & \\ & \nearrow A_{21} & & \searrow A_{42} & \\ 1 & & & & 4 \\ & \searrow A_{31} & & \nearrow A_{43} & \\ & & 3 & & \end{array} \tag{3}$$

The space $H_Q(\mathbf{n})$ of representations is

$$Hom(\mathbb{C}^{n_1}, \mathbb{C}^{n_2}) \oplus Hom(\mathbb{C}^{n_1}, \mathbb{C}^{n_3}) \oplus Hom(\mathbb{C}^{n_2}, \mathbb{C}^{n_4}) \oplus Hom(\mathbb{C}^{n_3}, \mathbb{C}^{n_4}).$$

$G_Q = GL(n_1) \times GL(n_2) \times GL(n_3) \times GL(n_4)$ acts on $H_Q(\mathbf{n})$ by

$$(g_1, g_2, g_3, g_4) \cdot (A_{21}, A_{31}, A_{42}, A_{43}) =$$

$$(g_2 A_{21} g_1^{-1}, g_3 A_{31} g_1^{-1}, g_4 A_{42} g_2^{-1}, g_4 A_{43} g_3^{-1})$$

Weights of Semi-invariants

Let Q be a quiver, $\mathbf{n} \in \mathbb{Z}_{\geq 0}^{Q_0}$ be a dimension vector **Let $H_Q(\mathbf{n})$ be the space of representations of Q of dimension n.**
$G = G_Q = \{(g_x), g_x \in GL(n_x)\}$.
$\sigma = \sigma_x$ a collection of integers : $\sigma \in \mathbb{Z}^{Q_0}$.
Let $\mathcal{P}_\sigma(\mathbf{n})$ be the space of semi-invariant polynomials functions p on $H_Q(\mathbf{n})$ of weight σ, that is

$$P(grg^{-1}) = \prod_x \det(g_x)^{-\sigma_x} P(r).$$

Definition : Let $m(\sigma, \mathbf{n})$ be the dimension of $\mathcal{P}_\sigma(\mathbf{n})$.
Definition : $\Sigma(\mathbf{n}) = \{\sigma, m(\sigma, \mathbf{n}) > 0\}$
Let N be any integer. It is easy to see that $m(\sigma, \mathbf{n}) > 0$ implies $m(N\sigma, \mathbf{n}) > 0$ Indeed if $P, P' \neq 0$ are semi-invariants of weight σ, σ', PP' is of weight $\sigma + \sigma'$.
We can also vary dimension vectors.
It is easy to see that if $m(\sigma, N\mathbf{n}) > 0$ then $m(\sigma, \mathbf{n}) > 0$

Questions

Q with no loops.
A : Can we describe $\Sigma(\mathbf{n})$, the set of elements $\sigma \in \mathbb{Z}^{Q_0}$ such that $m(\sigma, \mathbf{n}) > 0$ by explicit inequalities. (existence of "semi-stable" representations of Q)
B : Is it possible to answer the question $m(\sigma, \mathbf{n}) > 0$ in polynomial time.

Answer to **A :** yes ;
B : Open question. Some positive examples The Horn quiver, the quiver Q_4 *MV-MW*, the Kronecker quiver *Kac, Reineke*.

Subrepresentations and Inequalities

Let $Q = (Q_0, Q_1)$ be a quiver, and $\mathbf{n}$ be a dimension vector. Let $r = (r_a) : E_x \to E_y$ be a representation of Q, with $\dim E_x = n_x$. A subrepresentation of r is the data of $\mathcal{S} = (S_x, S_x \subseteq E_x)$, such that $r_a(S_x) \subseteq S_y$ for any $a : x \to y$. We write $r(\mathcal{S}) \subseteq \mathcal{S}$.

Notation : Let $0 \leq \alpha_x \leq n_x$ integers. We denote $\alpha \subseteq_Q \mathbf{n}$ if every representation r of Q of dimension vector $\mathbf{n}$ has a subrepresentation $\mathcal{S}$, with $\dim S_x = \alpha_x$

In other words, $H_Q(\mathbf{n})$ is the orbit by $GL(Q)$ of elements

$$r = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$$

with $r_{11} : \mathbb{C}^{\alpha_x} \to \mathbb{C}^{\alpha_y}$, $r_{22} : \mathbb{C}^{\beta_x} \to \mathbb{C}^{\beta_y}$.
Here $\alpha_x + \beta_x = n_x$. $\alpha_y + \beta_y = n_y$.

Subrepresentations and King's inequalities

If P is a nonzero semi-invariant polynomial with weight $\sigma = (\sigma_x)$, and $\alpha \subseteq_Q \mathbf{n}$, there exist r as above such that $P(r) \neq 0$. Consider $g = (g_x)$ with g_x in blocks $\alpha_x \times \alpha_x$, $\beta_x \times \beta_x$

$$g_x(t) = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}$$

By the semi-invariance condition :

$$P(g(t)rg(t)^{-1}) = e^{-t(\sum \sigma_x \alpha_x)} P(r)$$

But $g(t)rg(t)^{-1}$ has limit when $t \to -\infty$ the matrix $\begin{pmatrix} r_{11} & 0 \\ 0 & r_{22} \end{pmatrix}$. So

$$\sum \sigma_x \alpha_x \leq 0.$$

So we obtain inequalities associated to subrepresentations and their dimension vectors α_x.

Some Answers to Existence of semi-invariants

A : $m(\sigma, \mathbf{n}) > 0$ if and only if $\sum_x \sigma_x n_x = 0$ and $\sum_x \sigma_x \alpha_x \leq 0$ for all $\alpha \subseteq_Q \mathbf{n}$

Inequalities of A. King

B : $m(\sigma, \mathbf{n}) > 0$ if and only if $m(N\sigma, n) > 0$ for some $N > 0$

Derksen Weyman : Saturation property

We can also vary dimension vectors and obtain the remarkable property : **C :** $\Sigma(\mathbf{n}) = \Sigma(N\mathbf{n})$

Derksen Weyman : Duality

How to determine explicitly the elements $\alpha \subseteq_Q \mathbf{n}$

Euler form :

$$Euler(\alpha, \beta) = \sum_x \alpha_x \beta_x - \sum_{a:x \to y} \alpha_x \beta_y.$$

If $\alpha \subseteq_Q \mathbf{n}$, $\beta_x = n_x - \alpha_x$ then $Eul(\alpha, \beta) \geq 0$: (comes easily from $\{grg^{-1}, g \in GL_Q, r(\mathcal{S}) \subseteq \mathcal{S}\} = H_Q$)

Theorem

(Schoffield, V.-Walter)
$\alpha \subseteq_Q \mathbf{n}$ if and only if
1 $Euler(\alpha, \beta) \geq 0$ for $\beta = \mathbf{n} - \alpha$
2 if $\beta \subseteq_Q \alpha$ and $\beta \neq \alpha$, then $\beta \subseteq_Q \mathbf{n}$.

This condition be transformed to a numerical recurrence conditions $Euler(\alpha, \mathbf{n} - \beta) \geq 0$ for $\beta \subseteq_Q \alpha$ and we can minimize the possible $\beta \subseteq_Q \alpha$ to check by various conditions ;

Example : Q_4

Let us consider the dimension vector $\mathbf{n} = [2,2,2,3]$.
The following $\alpha' s$

$$[0,0,0,1],[1,1,2,2],[1,2,1,2]$$

are such that $\alpha \subseteq_Q \mathbf{n}$.
To describe if $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \Sigma_Q(\mathbf{n})$ we need to check Equality

$$2\sigma_1 + 2\sigma_2 + 2\sigma_3 + 3\sigma_4 = 0$$

King's Inequalities

$$\sigma_4 \leq 0, \ , \sigma_1 + \sigma_2 + 2\sigma_3 + 2\sigma_4 \leq 0, \ \sigma_1 + 2\sigma_2 + \sigma_3 + 2\sigma_4 \leq 0$$

Thus $\sigma = (1,1,1,-2)$ is a weight of a semi-invariant polynomial

$$P(A_{21}, A_{31}, A_{43}, A_{42}) = det \begin{pmatrix} I_3 & A_{42} & 0 & 0 \\ I_3 & 0 & A_{43} & 0 \\ I_3 & 0 & 0 & A_{43}A_{31} + A_{42}A_{21} \end{pmatrix}$$

where I_3 is the identity matrix of dimension 3 so this is 9 by 9 matrix.

Polytopes for multiplicities

Q quiver dimension vector $\mathbf{n} \in \mathbb{Z}_{\geq 0}^{Q_0}$. $H_Q(\mathbf{n})$ the corresponding space of representations . $GL_Q(\mathbf{n}) = \prod_x GL(n_x)$ acts on the space $Sym^*(H_Q(\mathbf{n}))$ of polynomial functions on $H_Q(\mathbf{n})$. $\vec{\lambda} = (\lambda_x)_{x \in Q_0}$ a sequence of highest weights for $Gl(n_x)$. $V_{\vec{\lambda}} = \otimes_x V_{\lambda_x}$.
Decompose

$$Sym^*(H_Q(\mathbf{n})) = \bigoplus_{\vec{\lambda}} m_Q(\vec{\lambda}) V^{\vec{\lambda}}$$

When Q is the quiver

$$1 \rightarrow 3 \leftarrow 2 \tag{4}$$

and the dimension vector is $\mathbf{n} = [n, n, n]$, then $m_Q(\lambda_1, \lambda_2, \lambda_3)$ is the multiplicity of the trivial representation of $GL(n)$ in $V^{\lambda_1} \otimes V^{\lambda_2} \otimes V^{\lambda_3}$, that is $C(\lambda_1, \lambda_2, \lambda_3) = c_{\lambda_1, \lambda_2}^{\lambda_3^*}$.
So multiplicities for quivers : generalisation of the Clebsch-Gordan coefficients.

Multiplicity function

A : The cone $Cone(Q, \mathbf{n}) = \{\vec{\lambda}; m_Q(\vec{\lambda}) > 0\}$ generates a rational polyhedral cone with Horn inequalities of the form

$$\sum_x \sum_{j \in I_x} \lambda_x(j) \leq 0$$

where $\mathcal{I} = (I_x)$, I_x subsets of $[1, 2, \ldots, n_x]$ determined by an induction relation. $\mathcal{I} \subseteq_{Q,B} \mathbf{n}$ *V-Walter, other proof Bertozzi-Reineke*
B The cone $Cone(Q, \mathbf{n})$ is saturated. Any integral point $\vec{\lambda}$ satisfying the Horn inequalities is such that $m_Q(\vec{\lambda}) > 0$. follows from Derksen-Weyman saturation)

Polytopes for multiplicity

The question to find a polytopal description for $m_Q(\vec{\lambda})$ was raised by Chindris-Collins-Kline, and solved for bipartite quivers with a beautiful polytope generalizing the Knutson-Tao Hive polytope.
We remarked :
for any quiver Q and dimension vector $\mathbf{n}$, there exists a family of polytopes $P_Q(\vec{\lambda})$ such that, when $\vec{\lambda}$ is a dominant weight, the number of integral points in $P_Q(\vec{\lambda})$ equals $m_Q(\vec{\lambda})$.
Moreover, $P_Q(\vec{\lambda})$ can be described by a combinatorial linear program that can be generated in strongly polynomial time given Q (given by the number of vertices and the list of arrows, encoded by pairs of integers) and $\vec{\lambda}$ (given by a list of integer vectors λ_x of size n_x) ; the right-hand side of the inequalities depend linearly on $\vec{\lambda}$ and all coefficients are in $\{0, 1, -1\}$.
So there exists a strongly polynomial time algorithm that decides membership in the moment cone when given as input a quiver Q and a dominant weight $\vec{\lambda}$.

An example

Consider the quiver Q_4, and (to simplify) $\mathbf{n} = (n, n, n, n)$.

$$\begin{array}{ccccc} & & 2 & & \\ & \nearrow & & \searrow & \\ 1 & & & & 4\,, \\ & \searrow & & \nearrow & \\ & & 3 & & \end{array} \tag{5}$$

$$m_Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

$$= \sum_{\alpha,\beta,\gamma,\delta \geq 0} C(\lambda_1^*, \alpha, \beta) C(\lambda_2^*, \alpha^*, \gamma) C(\lambda_3^*, \beta^*, \delta) C(\lambda_4^*, \gamma^*, \delta^*).$$

Follows from *Cauchy formula*

$$Sym^*(Hom(\mathbb{C}^n, \mathbb{C}^n)) = \oplus_{\mu \geq 0} V^\mu \otimes V^{(\mu)^*}$$

under $GL(n) \times GL(n)$.

A polytope

We can write this multiplicity as the number of integral points in the following polytope :

$$P_Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left\{ (\alpha, \beta, \gamma, \delta, p_1, p_2, p_3, p_4) \right\}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}^n_{\geq 0}$ satisfying Weyl inequalities,
$p_1 \in KT(\lambda_1^*, \alpha, \beta)$, $p_2 \in KT(\lambda_2^*, \alpha^*, \gamma)$, $p_3 \in KT(\lambda_3^*, \beta^*, \delta)$,
$p_4 \in KT(\lambda_4^*, \gamma^*, \delta^*)$.
Similar formulae for all quivers Q and any dimension vector.

A polytope for $m_{Q_4}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

Example $\mathbf{n} = [\mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}]$