TOWARDS LOWER BOUNDS ON ALGEBRAIC FORMULAS

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Algebraic Formulas

$$P(x_1, x_2, x_3) = 1 - 2x_1 + x_2 - x_3 - 2x_1x_2 - x_2x_3 + x_1x_3$$

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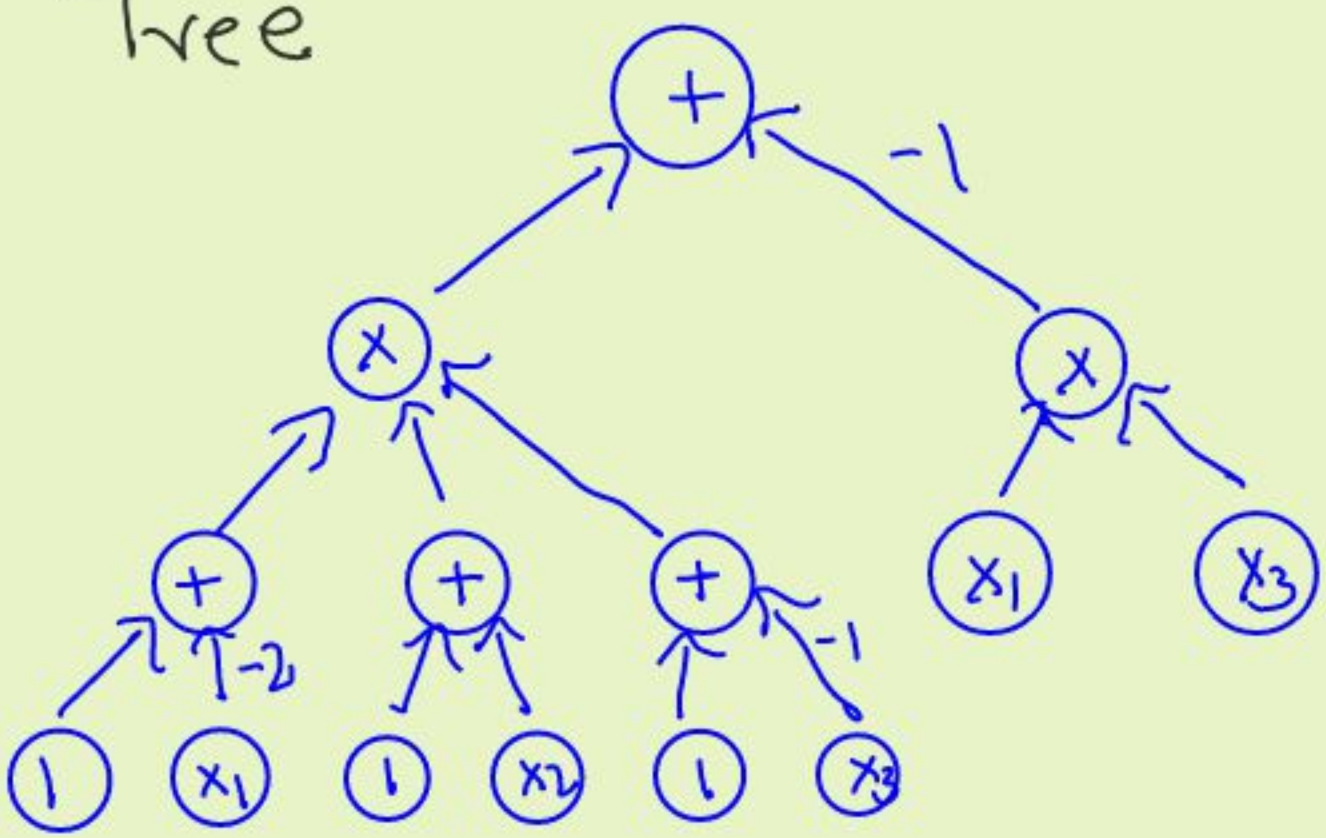
Tree



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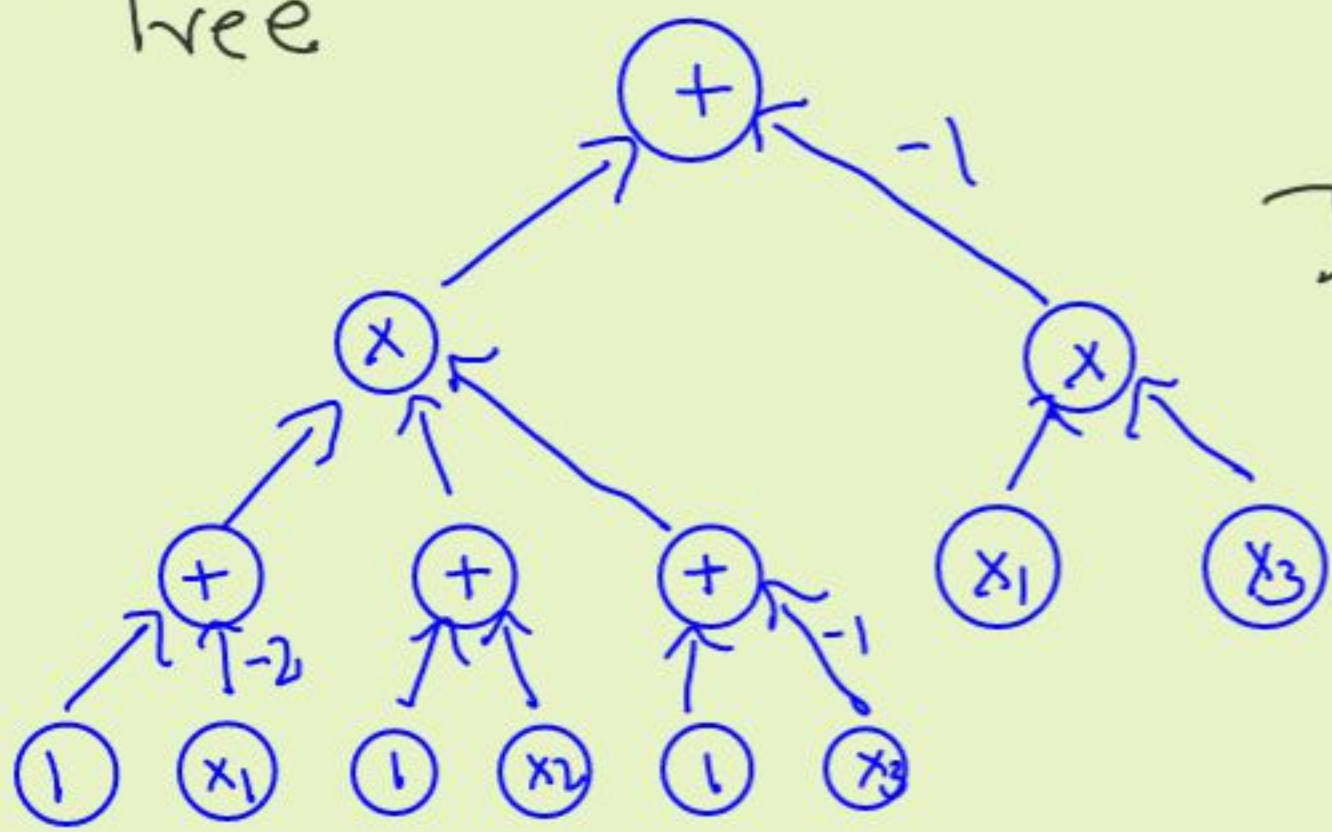


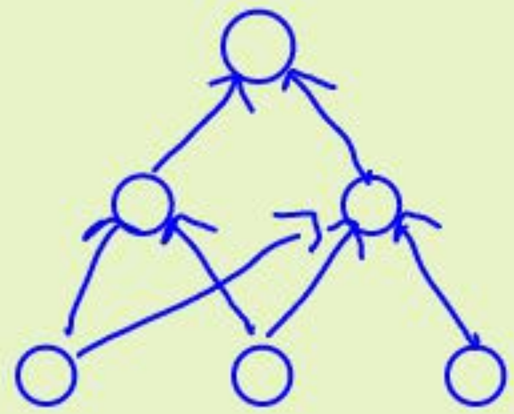
Size = # ops = 6

Depth = longest path = 3

Algebraic Circuits

Circuits

DAG



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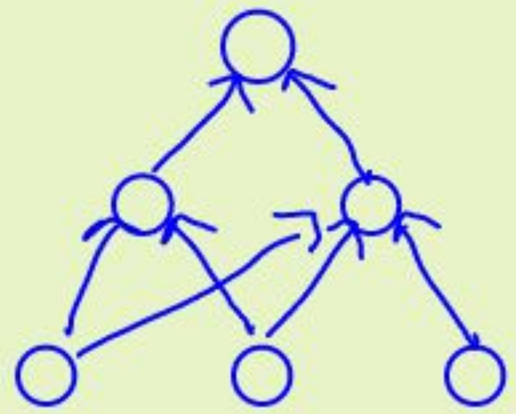

Trivial

Formulas of size poly(n, d) $\subseteq$ Ckts of size poly(n,d)

n → # vars

d → degree

Algebraic Circuits

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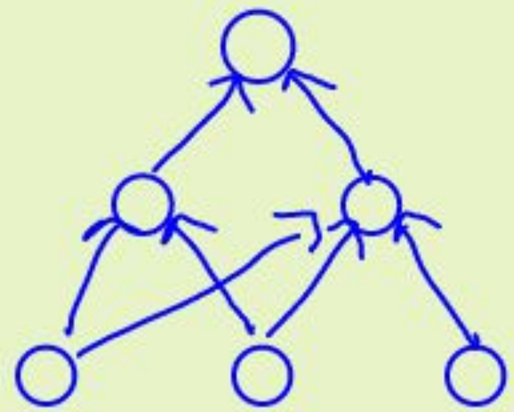
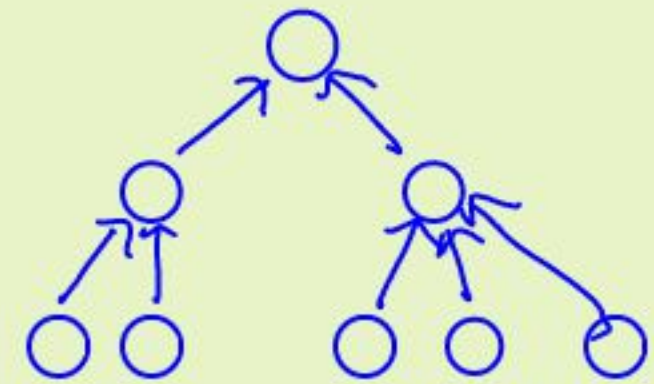


Formulas



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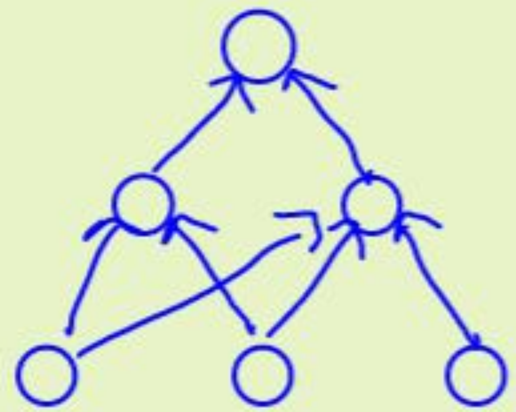
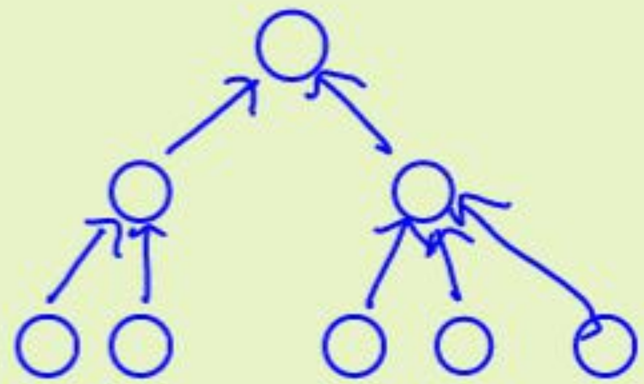


Formulas



[Hyafil '70s]

Formulas of size $\text{poly}(n, d)$ $\overset{\text{Trivial}}{\subseteq}$ Ckts of size $\text{poly}(n, d)$ $\subseteq$ Formulas of size $n^{O(\log d)}$

n → # vars

d → degree

$n^{O(\log d)}$ → quasipoly blowup

Lower bounds [Valiant'79]

Question: Find sequences of polynomials that have no formulas / circuits of poly(n, d) size?

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$$VNC^1 \underset{\neq ?}{\subseteq} VP \underset{\neq ?}{\subseteq} VNP$$

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$$\begin{array}{ccccc} VNC^1 & \underset{?}{\subsetneq} & VP & \underset{?}{\subsetneq} & VNP \\ NC^1 & \underset{?}{\subsetneq} & P & \underset{?}{\subsetneq} & NP \end{array}$$

→ Algebraic question easier [Bürgisser'00]

Syntactic vs. Semantic

Lower bounds [Valiant'79]

Question: Find sequences of polynomials that have no formulas / circuits of $poly(n,d)$ size?

Polynomial Identity Testing (PIT)

Input: $P(x_1, \ldots, x_n)$ (as formula/circuit)

Output: $P \stackrel{?}{\equiv} 0$

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→ Easy randomized algo. [O'22, DL'78, Z'79, S'80]

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→ Lower bounds ⇒ deterministic algos.

[KI'03, DSY'09, CKS'18, GKSS'19 ...]

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Formula question seems easier:

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Question: Find sequences of polynomials that have no formulas / circuits of $\mathrm{poly}(n,d)$ size?

Formula question seems easier:

→ Better results known: $\Omega(n^2)$ vs. $\Omega(n \lg d)$

[Kalorkoti '82] [Baur-Strassen'83]

Lower bounds [Valiant'79]

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→ Superpoly lower bds. in restricted settings (Multilinear [Raz'04, ...], Constant-depth)

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→ Superpoly lower bds. in restricted settings
(Multilinear [Raz'04, ...], Constant-depth)

→ Techniques available

Ben-Or Construction

$$e_d(x_1, \ldots, x_n) \doteq \sum_{S \subseteq [n] : |S| = d} \prod_{i \in S} x_i \quad - \binom{n}{d}$$

monomials

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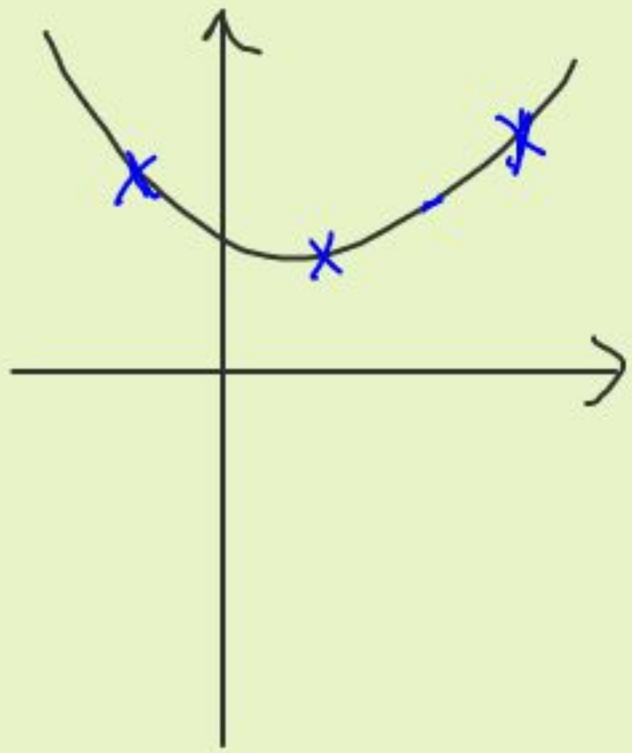
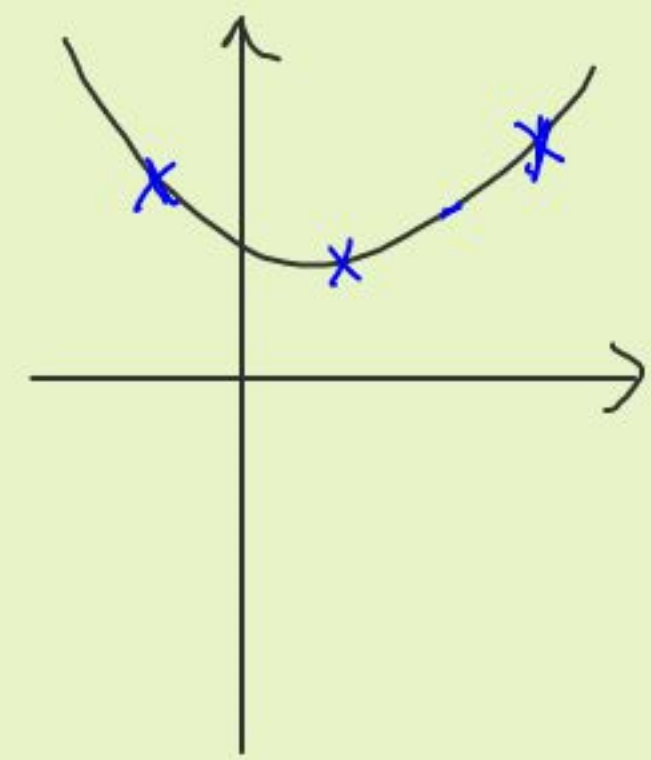


Ben-Or Construction

$$e_d(x_1,\dots,x_n) = \sum_{S\subseteq[n]:|S|=d} \prod_{i\in S} x_i \quad - \binom{n}{d}$$

monomials

$$U(t) = \prod_{i=1}^{n}(1+tx_i) = \sum_{j=0}^{n} t^j \cdot e_j$$



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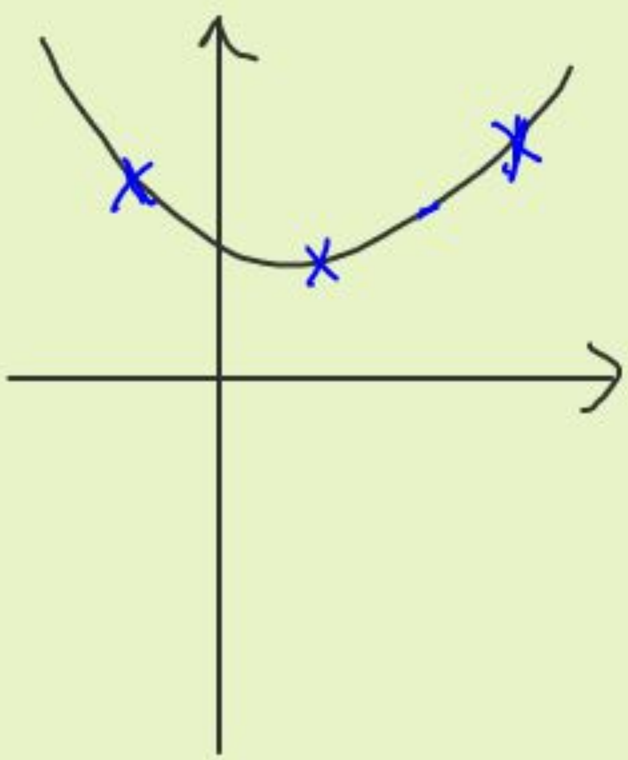
$$e_d(x_1, \ldots, x_n) \doteq \sum_{S \subseteq [n] : |S| = d} \prod_{i \in S} x_i \quad - \binom{n}{d}$$

monomials

$$U(t) = \prod_{i=1}^{n} (1 + t x_i)$$

$$= \sum_{j=0}^{n} t^j \cdot e_j$$



$$e_d = \sum_{i=0}^{n} \beta_i \, U(\alpha_i)$$

$$= \sum_{i=0}^{n} \beta_i \prod_{j=1}^{n} (1 + \alpha_i x_j)$$

Ben-Or Construction

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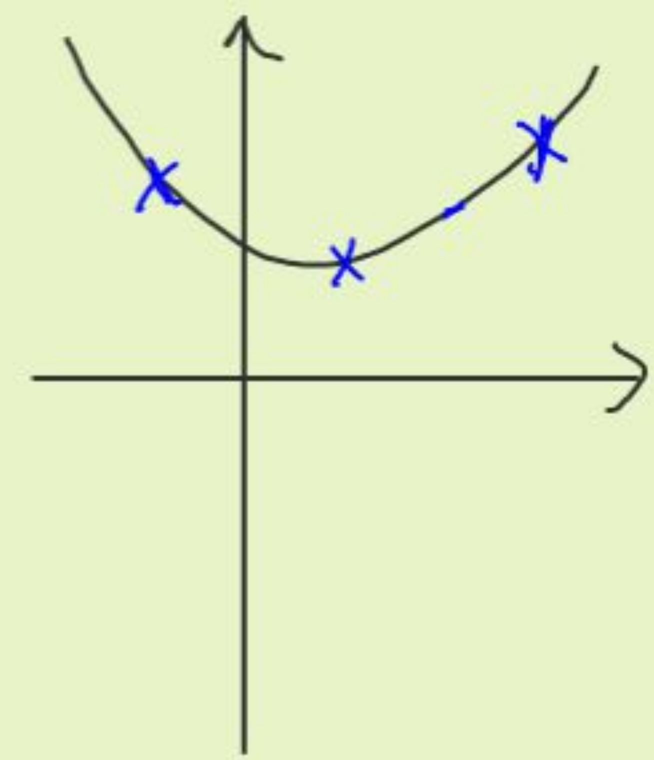

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Depth-3 formula of size $O(n^2)$

Suspected Hard Polynomial: IMM

$$\begin{pmatrix} & & \\ & X_1 & \\ & & \end{pmatrix}_{n\times n} \begin{pmatrix} & & \\ & X_2 & \\ & & \end{pmatrix} \cdots \begin{pmatrix} & & \\ & X_d & \\ & & \end{pmatrix} = \begin{pmatrix} \Box \\ \downarrow \\ IMM_{n,d} \end{pmatrix}$$

Suspected Hard Polynomial: IMM

$$\begin{pmatrix} & X_1 & \end{pmatrix}_{n\times n}\begin{pmatrix} & X_2 & \end{pmatrix}\cdots\begin{pmatrix} & X_d & \end{pmatrix} = \begin{pmatrix} \square & \\ \downarrow & \\ IMM_{nd} & \end{pmatrix}$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} x_{11}y_{11}z_{11} + x_{11}y_{12}z_{21} & \cdots \\ + x_{12}y_{22}z_{21} + x_{12}y_{21}z_{11} & \\ & \cdots \\ \cdots & \end{pmatrix}$$

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Fact: $IMM_{n,d} \in VP$

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Fact: $IMM_{n,d} \in VP$

Conjecture: $IMM_{n,d}$ has no formulas of size $poly(n,d)$.

["Correct" answer $\approx n^{\log d}$]

High-level approach to formula lower bounds

Step 1:

Step2 ;

High-level approach to formula lower bounds

Step 1: Wlog, algebraic formulas for $IMM_{n,d}$ have additional syntactic structure.

[Combinatorial]

Step 2:

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Step 1: Wlog, algebraic formulas for $IMM_{n,d}$ have additional syntactic structure.

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Step 2: Show that structured formulas for $IMM_{n,d}$ have large size

[Algebraic]

Multilinear formulas

$$\begin{pmatrix} & X_1 & \end{pmatrix}_{n\times n} \begin{pmatrix} & X_2 & \end{pmatrix} \cdots \begin{pmatrix} & X_d & \end{pmatrix} = \begin{pmatrix} \square \\ \downarrow \\ IMM_{nd} \end{pmatrix}$$

Multilinear polynomial : No variable appears twice in a monomial.

Multilinear formulas

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Multilinear polynomial : No variable appears twice in a monomial.

Multilinear formula : All intermediate computations are multilinear.

Multilinear formulas

$$\underbrace{\left(X_1 \right)}_{n\times n} \left(X_2 \right) \cdots \left(X_d \right) = \left(\begin{matrix} \square \\ \downarrow \\ IMM_{nd} \end{matrix} \right)$$

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$$\underbrace{X_1 \cdot X_2}_{\text{Multilinear formula}}$$

Multilinear formulas

$$\begin{pmatrix} & X_1 & \end{pmatrix}_{n\times n} \begin{pmatrix} & X_2 & \end{pmatrix} \cdots \begin{pmatrix} & X_d & \end{pmatrix} = \begin{pmatrix} \square \\ \downarrow \\ \mathrm{IMM}_{nd} \end{pmatrix}$$

Multilinear polynomial : No variable appears twice in a monomial.

Multilinear formula : All intermediate computations are multilinear.

$$\underbrace{x_1 \cdot x_2}_{\text{Multilinear formula}} = \underbrace{\frac{1}{2}\cdot(x_1+x_2)^2 - \frac{1}{2}\cdot(x_1^2+x_2^2)}_{\text{Non-multilinear formula}}$$

Lower bounds for multilinear formulas

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$P(x_1, \ldots, x_n)$ multilinear

$\{x_1, \ldots, x_n\} = Y \sqcup Z \quad (|Y| = |Z| = n/2)$

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$$M_P = \begin{array}{c|c} & m_2(Z) \\ \hline m_1(Y) & \mathrm{Coeff}_P(m_1 m_2) \end{array}$$

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Eg: $P = x_1 x_2 + x_3 x_4$

$Y = \{x_1, x_3\}$

$Z = \{x_2, x_4\}$

$$M_P = \begin{array}{c} \\ 1 \\ x_1 \\ x_3 \\ x_1 x_3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x_2 & x_4 & x_2 x_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

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Thm [NW'95, Raz'04]: P has small ml formula $\Rightarrow$ $rk(M_P)$ small.

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Lower bounds on multilinear formulas

Thm [Raz'04]: No poly-sized ml formulas for the Determinant.

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Thm [RY'08]: Stronger lower bounds on constant-depth ml formulas.

Thm [DMPY'12]: Lower bounds for variants of $IMM_{n,n}$.

$$\begin{pmatrix} & X_1 & \end{pmatrix}_{n\times n} \begin{pmatrix} & X_2 & \end{pmatrix} \cdots \begin{pmatrix} & X_d & \end{pmatrix} = \begin{pmatrix} \boxed{} \\ \downarrow \\ IMM_{nd} \end{pmatrix}$$

Structural Results

P has a small algebraic formula [SY'10] $\overset{?}{\Longrightarrow}$ P has a small ml formula

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Thm [CLS'18]: For every constant Δ, there is a P_Δ that has small formulas of depth Δ but no small ml formulas of depth Δ.

Structural Results

P has a small algebraic formula $\overset{?}{\Longrightarrow}$ P has a small ml formula [SY'10]

Thm [CLS'18]: For every constant Δ, there is a P_Δ that has small formulas of depth Δ but no small ml formulas of depth Δ.

Moral: Multilinearization seems hard?

Towards Algebraic formula lower bounds

Step 1: Wlog, algebraic formulas for $IMM_{n,d}$ have additional syntactic structure.
[Combinatorial]

Step 2: Show that structured formulas for $IMM_{n,d}$ have large size
[Algebraic]

Towards Algebraic formula lower bounds

	Step 1	Step 2
Multilinear formulas	?	✓

Step 1: Wlog, algebraic formulas for $IMM_{n,d}$ have additional syntactic structure.
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Set-Multilinear formulas

$$\Big(\; X_1 \; \Big)_{n \times n} \Big(\; X_2 \; \Big) \cdots \Big(\; X_d \Big) = \begin{pmatrix} \square \\ \downarrow \\ \mathrm{IMM}_{nd} \end{pmatrix}$$

Set-Multilinear polynomial : No matrix contributes twice to a monomial

Set-multi-linear formula : All intermediate computations are set-multilinear

Structural results

Structural results

Thm [Raz '10]: $d \ll \log n$

$IMM_{n,d}$ has small algebraic formula $\Rightarrow$ $IMM_{n,d}$ has small s.m. formula.

Structural results

Step 2 is more challenging.

Thm [Raz '10]: $d \ll \log n$

$\mathrm{IMM}_{n,d}$ has small algebraic formula $\Rightarrow$ $\mathrm{IMM}_{n,d}$ has small s.m. formula.

Structural results

Step 2 is more challenging.

Thm [Raz '10]: $d \ll \log n$

$IMM_{n,d}$ has small algebraic formula $\Rightarrow$ $IMM_{n,d}$ has small s.m. formula.

Thm [LST'21, Forbes'24]: $d \ll \log n$

$IMM_{n,d}$ has small algebraic formula of depth Δ $\Rightarrow$ $IMM_{n,d}$ has small s.m. formula of depth 2Δ

Lower bounds on s.m.l. formulas

Thm [LST'21, Forbes'24]: No poly-sized s.m.l formula for $IMM_{n,d}$ of constant depth for any $d(n)$.

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Cor: No poly-sized formula for $IMM_{n,d}$ of constant depth.

Lower bounds on s.m.l. formulas

Thm [LST'21, Forbes'24]: No poly-sized s.m.l formula for $IMM_{n,d}$ of constant depth for any $d(n)$.

Cor: No poly-sized formula for $IMM_{n,d}$ of $\underbrace{\text{constant depth.}}_{\text{works until depth } \log\log d}$

Lower bounds on s.m.l. formulas (any depth)

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Thm [FLMST'23]:

Small s.m.l. formulas for $\mathrm{IMM}_{n,d}$ $\Rightarrow$ Small s.m.l. formulas of depth $O(\log d)$

Lower bounds on s.m.l. formulas (any depth)

Thm [FLMST'23]:

Small s.m.l. formulas for $\mathrm{IMM}_{n,d}$ $\Rightarrow$ Small s.m.l. formulas of depth $O(\log d)$

Thm [TLS'22]:

$\underbrace{\text{No poly-sized}}_{n^{\Omega(\log\log n)}}$ s.m.l. formula for $\mathrm{IMM}_{n,n}$

Lower bounds on s.m.l. formulas (any depth)

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Small s.m.l. formulas for $IMM_{n,d}$ $\Rightarrow$ Small s.m.l. formulas of depth $O(\log d)$

<u>Thm</u> [TLS'22]:

No poly-sized s.m.l. formula for $IMM_{n,n}$

$\underbrace{\text{No poly-sized}}_{n^{\Omega(\log\log n)}} \xrightarrow{?} n^{\Omega(\log n)}$

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$\Downarrow$ [TLS'22]

No poly-sized formula for $\mathrm{IMM}_{n,n}$.

Towards Algebraic formula lower bounds

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Multilinear formulas	?	✓

<u>Step 1</u>: Wlog, algebraic formulas for $IMM_{n,d}$ have additional syntactic structure.
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<u>Step 2</u>: Show that structured formulas for $IMM_{n,d}$ have large size
[Algebraic]

Towards Algebraic formula lower bounds

	Step 1	Step 2
Multilinear formulas	?	✓
S.m.l. formulas ($d \ll \log n$)	✓	?

Step 1: Wlog, algebraic formulas for $IMM_{n,d}$ have additional syntactic structure. [Combinatorial]

Step 2: Show that structured formulas for $IMM_{n,d}$ have large size [Algebraic]

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Multilinear formulas	?	✓
S.m.l. formulas ($d \ll \log n$)	✓	?
s.m.l formulas ($\Delta = O(1)$)	✓	✓

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	Step 1	Step 2
Multilinear formulas	?	✓
S.m.l. formulas ($d \ll \log n$)	✓	?
s.m.l formulas ($\Delta = O(1)$)	✓	✓
S.m.l. ($d > \log n$)	?	✓

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Input : $P(x_1, \ldots, x_n)$ (as formula / circuit)

Output : $P \stackrel{?}{\equiv} 0$

Lower bounds $\Longrightarrow$ deterministic PIT

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Input : $P(x_1, \ldots, x_n)$ (as formula / circuit)

Output : $P \stackrel{?}{\equiv} 0$

Lower bounds $\Longrightarrow$ deterministic PIT

Cor [CKS'18 + LST'21]: Deterministic sub-exp.-time PIT algorithms for constant-depth formulas.

Homogeneous formulas

$$\begin{pmatrix} & X_1 & \end{pmatrix}_{n\times n} \begin{pmatrix} & X_2 & \end{pmatrix} \cdots \begin{pmatrix} & X_d & \end{pmatrix} = \begin{pmatrix} \square \\ \downarrow \\ IMM_{nd} \end{pmatrix}$$

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Homogeneous formula: All intermediate computations are homogeneous.

Ben-Or construction: Inhomogeneous formula.

Structural Results

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Cor: "Correct" homogeneous lower bd. for $IMM_{n,d}$ ⟹ General formula lower bound.

Optimal homogenization?

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$X_1, \dots, X_n$ - 3×3 matrices of variables.

$$ME_d(X_1, \dots, X_n) = \Big[\sum_{i_1 < \cdots < i_d} X_{i_1} \cdots X_{i_d} \Big]_{1,1}$$

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Homogenization with polynomial overhead $\overset{[\text{DGIJL'24}]}{\Longleftrightarrow}$ ME_d has homogeneous formulas of size $\text{poly}(n,d)$.

Thm [FLST'24]: "Non-commutative" formulas for e_d require superpolynomial size.

Lower bounds for homogeneous formulas

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Thm [FLST'24]: Lower bounds for weighted homogeneous formulas.

Towards Algebraic formula lower bounds

	Step 1	Step 2
Multilinear formulas	?	✓
S.m.l. formulas ($d \ll \log n$)	✓	?
s.m.l formulas ($\Delta = O(1)$)	✓	✓
S.m.l. ($d > \log n$)	?	✓

Step 1: Wlog, algebraic formulas for $IMM_{n,d}$ have additional syntactic structure. [Combinatorial]

Step 2: Show that structured formulas for $IMM_{n,d}$ have large size [Algebraic]

Towards Algebraic formula lower bounds

	Step 1	Step 2
Multilinear formulas	?	✓
S.m.l. formulas ($d \ll \log n$)	✓	?
S.m.l formulas ($\Delta = O(1)$)	✓	✓
S.m.l. ($d > \log n$)	?	✓
Homog. ($d < n^{O(1)}$)	......	?

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Towards Algebraic formula lower bounds

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Multilinear formulas	?	✓
S.m.l. formulas ($d \ll \log n$)	✓	?
s.m.l formulas ($\Delta = O(1)$)	✓	✓
	?	✓
S.m.l. ($d > \log n$)		?
Homog. ($d < n^{o(1)}$)	✓ (dotted)	
Homog. ($d < \log n$, $\Delta = O(1)$)	✓	✓

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Multilinear formulas	?	✓
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S.m.l formulas ($\Delta = O(1)$)	✓	✓
S.m.l. ($d > \log n$)	?	✓
Homog. ($d < n^{o(1)}$)	✓	?
Homog. ($d < \log n$, $\Delta = O(1)$)	✓	✓
Weighted homogeneous	?	✓

Step 1: Wlog, algebraic formulas for $IMM_{n,d}$ have additional syntactic structure.
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What I didn't cover

- Set-multilinear formula lower bounds
  [KS'22, KS'23]

- Algebraic Branching Programs
  [RSY'08, CKSV'22, CKSV'24]

- Poly-time & Quasipoly time PIT algorithms
  [DDS'21, PS'21]

- ...

Towards Algebraic formula lower bounds

	Step 1	Step 2
Multilinear formulas	?	✓
S.m.l. formulas ($d \ll \log n$)	✓	?
s.m.l formulas ($\Delta = O(1)$)	✓	✓
S.m.l. ($d > \log n$)	?	✓
Homog. ($d < n^{o(1)}$)	✓ (dotted)	?
Homog. ($d < \log n$, $\Delta = O(1)$)	✓	✓
Weighted homogeneous	?	✓

Step 1: Wlog, algebraic formulas for $IMM_{n,d}$ have additional syntactic structure. [Combinatorial]

Step 2: Show that structured formulas for $IMM_{n,d}$ have large size [Algebraic]

$? \xrightarrow{?} ✓$

$✓ \xrightarrow{?} ✓$

d small $\xrightarrow{?}$ all d

Towards Algebraic formula lower bounds

	Step 1	Step 2	
Multilinear formulas	?	✓	
S.m.l. formulas ($d \ll \log n$)	✓	?	
s.m.l formulas ($\Delta = O(1)$)	✓	✓	
S.m.l. ($d > \log n$)	?	✓	? $\xrightarrow{?}$ ✓
Homog. ($d < n^{o(1)}$)	✓	?	✓ $\xrightarrow{?}$ ✓
Homog. ($d < \log n$, $\Delta = O(1)$)	✓	✓	d small $\xrightarrow{?}$ all d
Weighted homogeneous	?	✓	

Step 1: Wlog, algebraic formulas for $IMM_{n,d}$ have additional syntactic structure. [Combinatorial]

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Thanks!