Stabilizer Limits and Alignment - Lie Algebraic Methods for the Orbit Closure Problem.

Bharat Adsul, Milind Sohoni - IIT Bombay
K V Subrahmanyam - CMI, Chennai

Key Reference: *Orbit Closures, Stabilizer Limits and Intermediate G-varieties*, arxiv 2309.15816.
Also: *Lie Algebraic Methods for Orbit Closures*, arxiv 2201.00135.

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Model agnostic !

1st April 2025, WACT 2025, Ruhr University, Bochum

- Introduction - det_n as the master function and GCT
- The Geometric Approach - λ and the tangent of approach - Stabilizer Limits, Theorem 1 and the question of "genericity"
- Alignment, the classical $P(\lambda), U(\lambda)$ and Theorem 2 - alignment or nilpotency
- Determinant as the master group and compactification of $\mathcal{K}_n$
- Overall...
- Connecting with classical GIT limits - what holds and what are its analogues
- Other work in progress

Outline

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Determinant: The master function and the 1-PS

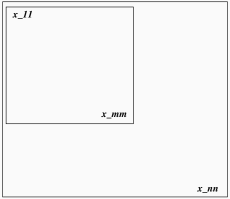
- Let $X_n = \mathbb{C}^{n \times n}$ be an $n \times n$-matrix of indeterminates, and let $det_n(X_n) \in Sym^n(X_n)$ be the usual determinant.
- Valiant. Let $X_m \subseteq X_n$ and $f \in Sym^m(X_m)$. Suppose that f has a formula of size n/c (where c is a constant) then there a linear map $A : X_m \to X_n$ such that $f'_A = x_{nn}^{n-m} f = det(AX_m)$.
- The form f'_A is called the padded form. The smallest n is called the determinantal complexity of f. We call A as the implementation of f as a determinant.

Valiant's Result

Establishes det_n as a master function for coding computations.

The Permanent as Determinant Question

- $X_m = \mathbb{C}^{m \times m}$, $f = perm_m(X_m)$. Let $V = Sym^n(X)$ and $y = det_n(X)$, $z = x_{nn}^{n-m} perm_m(X_m) \in V$.
- Algebraic Question: What is the smallest n such that $det_n(AX) = x_{nn}^{n-m} perm_m(X_m)$ (where $A \in M_n$).
- Both y, z have large stabilizers in $GL(X)$..but so far, no direct connection between stabilizers!



$K = K_n =$ Stabilizer of det_n in $GL(X_n)$

- $X_n \to CX_nD$ such that $C, D \in GL_n$ and $det_n(CD) = 1$ and $X \to X^T$.
- $K = G_y$ is reductive, $dim(G_y) = 2n^2 - 2$ and X_n is an irreducible G_y-module.

Stabilizers

H_m = Stabilizer of $z' = perm_m(X_m)$ in $GL(X_m)$

- $X_m \to CX_mD$ such that $C, D \in D_m$ and $det_m(CD) = 1$ and $X \to PX^TP'$, with P, P' permutation matrices.
- $G_{z'}$ is reductive, $dim(G_{z'}) = 2m - 2$ and X_m is an irreducible H_m-module.

$H = H_{n,m}$ = The stabilizer of the homogenized permanent

$z = x_{nn}^{n-m} perm_m(X_m) \in Sym^n(X_n)$. We may partition $X_n = \overline{X'_m} \oplus \mathbb{C}x_{nn} \oplus X_m \cong X_1 \oplus X_0$. Then $H_{n,m} = G_z \subseteq GL(X_n)$ in the ordered basis is as below:

$$\left[\begin{array}{c|c|c} * & * & * \\ \hline 0 & * & 0 \\ \hline 0 & 0 & g \end{array}\right] \text{ with } g \in tH_m$$

The Geometric Complexity Approach

- The stabilizers of forms are consequential in their computational complexity and universality.
- An algebraic framework based on Geometric Invariant Theory (GIT) to approach the problem - The orbit closure problem.
- Stabilizer data enough to determine if $z = x_{nn}^{n-m} perm_m$ may be obtained as a limit of $y = det_n$.
- Two key entry points:
  - Algebraic Consequence of Valiant's construction:There is a 2-block 1-PS $\lambda_A(t) \subseteq GL(X_n)$ such that:

    $$\lambda_A(t) det_n = x_{nn}^{n-m} perm_m + \sum_{i>0} t^i y_i$$

    where λ_A has weight spaces $X = X_0 \oplus X_1$.
  - Key GIT properties - stability of y, z', partial (or $L(\lambda)$) stability of z.

GIT Notation

- X over $\mathbb{C}$. $G \subseteq GL(X)$, connected reductive *algebraic* group over $\mathbb{C}$. Typically $G = GL(X)$.
- $\rho : GL(X) \to GL(V)$, $\mathbb{C}[V]$ ring of polynomial functions. Think $V = Sym^d(X^*)$.
- ρ representation such that the center $Z_{GL(X)} = \{tI | t \in \mathbb{C}^*\}$ acts as $\rho(tI)(v) = t^d v$ for a fixed d. Moreover, $Z_{GL(X)} \subseteq G$.
- For $y \in V$, *Orbit*, $O(y) := \{g \cdot y | g \in G\}$. $O(y)$ need not be closed, it is constructible.
- $\overline{O(y)}$, *orbit closure* of y - *Zariski* topology or *Euclidean topology*. $\overline{O(y)}$ is a cone and its ideal $I(y) \subseteq \mathbb{C}[V]$ is homogeneous.

Key Example: $GL(X_n)$ acting on $V = Sym^n(X_n)$, with $y, z \in V$.
Note that $det_n, perm_m$ are (resp.) SL_n-stable and SL_m-stable.

The 1-PS and orbit closure

Let $\lambda(t) \subseteq G$ be a 1-PS and suppose we have:

$$\lambda(t)y = t^d z + \ldots + t^D y_D \quad \text{(Notation: } y \xrightarrow{\lambda} z\text{)}$$

Recall $Z_{GL(X)} \subseteq G$. By applying a suitable power $t^a I$, we have:

$$\lambda'(t)y = t^0 z + \ldots + t^{D'} y_{D'}$$

Thus $z \in \overline{O(y)}$. Applying this to λ_A and $y = det_n$, and $z = x_{nn}^{n-m} perm_m$, we see that $z \in \overline{O(y)}$. That's the orbit closure.

Problem of Existence of $A \Rightarrow$ The Orbit Closure Problem

λ, y, z

- Given $z, y \in V$, is $z \in \overline{O(y)}$? Distinctive stabilizers, G_z, G_y.
- What connects $K = G_y$ and $H = G_z$ when $y \xrightarrow{\lambda} z$?

GCT and Representations as Obstructions

- Let $Y = \overline{O(y)}$ and $Z = \overline{O(z)}$, and $\mathbb{C}[Y] = \sum_\mu d_\mu V_\mu$ and $\mathbb{C}[Z] = \sum_\mu p_\mu V_\mu$ be their coordinate rings as G-modules.
- Stability of $det_n, perm_m$ and Peter-Weyl determine exactly which G-modules V_u appear in $\mathbb{C}[Y]$ and $\mathbb{C}[Z]$.
- $Z \subseteq Y \Rightarrow \mathbb{C}[Y] \twoheadrightarrow \mathbb{C}[Z]$ and thus $d_\mu \geq p_\mu$ for all μ.

GCT-II Conjecture

If $z \notin Y$ then there is a μ such that $p_\mu > 0$ and $d_\mu = 0$.

And its failure...

All V_μ which appear in $\mathbb{C}[Z]$, or for that matter, for the coordinate ring $\mathbb{C}[W]$ of the orbit closure $\overline{O(w)}$ of any homogenized form w, appear in $\mathbb{C}[Y]$.

So the numbers do matter.

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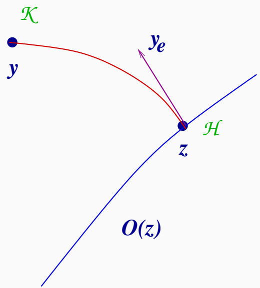
Our work - more geometric

We begin with:

$$y(t) = \lambda(t).y = y_d t^d + y_e t^e + \sum_{i=e+1}^{D} y_i t^i$$

with $z = y_d$. We call y_e as the tangent of approach.
We use the notation $y \xrightarrow{\lambda} z, z \xleftarrow{\lambda} y$ or $z = \widehat{y}^{\lambda}$ or simply $z = \hat{y}$.



Transversality. Vector space spanned by $y_e, \ldots, y_D$ intersects $T_g O(g)$ trivially.
Let $\mathcal{G} = Lie(G)$ and $\mathcal{K} = Lie(K)$, $\mathcal{H} = Lie(H) \subseteq \mathcal{G}$. These are infinitesimal group elements with the Lie bracket.

Question

How do we connect $\mathcal{K}$ with $\mathcal{H}$ using λ?

- $G \subseteq GL(X)$ of dimension r associated with a linear space $Lie(G) = \mathcal{G}$ of matrices of the same dimension.
- These are closed under the Lie bracket - $\mathfrak{a}, \mathfrak{b} \in \mathcal{G}$ then so is $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a}$.
- For an $\mathfrak{g} \in \mathcal{G}$, the family $exp(\mathfrak{g}, t) = e^{\mathfrak{g}t} \subset G$ is a curve with tangent $\mathfrak{g}$ at e.
- Finally, for $g \in G, \mathfrak{g} \in \mathcal{G}$, we have $g\mathfrak{g}g^{-1} \in \mathcal{G}$ as well, and is a group action.

Functorial equivalence between connected matrix groups and matrix Lie algebras and their modules.

Group	*Lie Algebra*
GL_m	$M_m = \mathbb{C}^{m \times m}$
SL_m	$\{m \in M_m, tr(m) = 0\}$
D_n	$\mathbb{C}^n$(diagonal matrices)
O_n	$\{m \in M_m \| m + m^T = 0\}$

- We have the usual action of λ on V and the weight space decomposition $V = \oplus V_i$. $\lambda(t)v = \sum_i t^i v_i$ with $v_i \in V_i$.
- $\lambda(t)$ also acts on $\mathcal{G}$ by conjugation and thus we have $\mathcal{G} = \oplus \mathcal{G}_i$.
- For any $v \in V, v = \sum_i v_i$, let the **leading term** $\widehat{v}^\lambda$ or simply $\hat{v}$ be v_j where $v_j \neq 0$ and $v_i = 0$ for all $i < j$. Similarly, we define $\widehat{\mathfrak{g}}^\lambda$ or simply $\hat{\mathfrak{g}}$ for any $\mathfrak{g} \in \mathcal{G}$.

Basic result: For any $\mathfrak{g} \in \mathcal{G}$ and $v \in V$:

Either $\hat{\mathfrak{g}}\hat{v} = 0$ or $\widehat{\mathfrak{g}v} = \hat{\mathfrak{g}}\hat{v}$ and $deg(\mathfrak{g}v) = deg(v) + deg(\mathfrak{g})$.

$$\begin{aligned}\lambda(t)(\mathfrak{g}v) &= (\lambda(t)\mathfrak{g}\lambda^{-1}(t))(\lambda(t)v) = \mathfrak{g}(t)v(t) \\ &= (\textstyle\sum_{i=a}^{A} t^i \mathfrak{g}_i)(\sum_{j=b}^{B} t^j v_j) = t^{a+b}\mathfrak{g}_a v_b + \ldots\end{aligned}$$

Then either $deg(\widehat{\mathfrak{g}v}) = a + b$ and then $\widehat{\mathfrak{g}v} = \hat{\mathfrak{g}}\hat{v}$, or $deg(\widehat{\mathfrak{g}v}) < a + b$, and then $\hat{\mathfrak{g}}\hat{v} = 0$.

Leading term algebras and modules

Proposition

Let $\mathcal{K}$ be a Lie subalgebra of $\mathcal{G}$ and $M \subseteq V$ a $\mathcal{K}$-module. Let $\hat{\mathcal{K}}$ (resp. $\hat{M}$) be the vector space generated by leading terms. Then
(i) $\hat{\mathcal{K}}$ is a graded Lie subalgebra of $\mathcal{G}$, and $dim_{\mathbb{C}}(\hat{\mathcal{K}}) = dim_{\mathbb{C}}(\mathcal{K})$,
(ii) $\hat{M} \subseteq V$ is a $\hat{\mathcal{K}}$-module with $dim_{\mathbb{C}}\hat{M} = dim_{\mathbb{C}}M$.

$$\begin{aligned} \lambda(t)([\mathfrak{k}, \mathfrak{k}']) &= [\lambda(t)\mathfrak{k}, \lambda(t)\mathfrak{k}'] \\ &= [\textstyle\sum_{i=a}^{A} t^i \mathfrak{k}_i, \sum_{j=b}^{B} t^j \mathfrak{k}'_j] = t^{a+b}[\mathfrak{k}_a, \mathfrak{k}'_b] + \dots \end{aligned}$$

Thus either $\widehat{[\mathfrak{k}, \mathfrak{k}']} = [\hat{\mathfrak{k}}, \hat{\mathfrak{k}}']$ or $[\hat{\mathfrak{k}}, \hat{\mathfrak{k}}'] = 0$. This proves that $\hat{\mathcal{K}}$ is a Lie algebra. That $\hat{M}$ is a $\hat{\mathcal{K}}$-module is clear from the last lemma. The dimension assertion is delicate but easily proved.

Corollary

If $m \in M$ and $\mathfrak{k} \in \mathcal{K}$ such that $\mathfrak{k}m = 0$, then $\hat{\mathfrak{k}}\hat{m} = 0$.

Example: Grading and Pose.

$G = GL_4(\mathbb{C})$, X a G-module, $y \in X$ and $\mathcal{K}$ as below. Let $\lambda(t)$ be as below.

$$\mathcal{K} = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \quad \lambda(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{bmatrix},$$

We see that:

$$\lambda(t) M \lambda(t)^{-1} = \begin{bmatrix} M_{11} & t^{-1} M_{12} \\ t M_{21} & M_{22} \end{bmatrix}$$

$\mathcal{K}$ commutes with λ and therefore $\hat{\mathcal{K}} = \mathcal{K}$, is reductive. $\hat{\mathcal{K}}_0 = \hat{\mathcal{K}}$.

Thus the dimension vector is

weight	-1	0	1
dimension	0	4	0

Pose matters... (2)

Let $\mathcal{K}' = A\mathcal{K}A^{-1}$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathcal{K}' = \begin{bmatrix} a & b & c & d-a \\ c & d & 0 & -c \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

If $\mathcal{K}'(t) = \lambda(t)\mathcal{K}'\lambda(t)^{-1}$, and so:

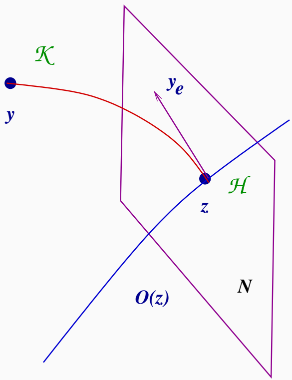
$$\mathcal{K}'(t) = \begin{bmatrix} a & b & t^{-1}c & t^{-1}(d-a) \\ c & d & 0 & -t^{-1}c \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \quad \widehat{\mathcal{K}}' = \begin{bmatrix} u & t & s & r \\ 0 & u & 0 & -s \\ 0 & 0 & u & t \\ 0 & 0 & 0 & u \end{bmatrix}$$

$\hat{\mathcal{K}}'$ is a solvable Lie algebra.

The dimension vector is

weight	-1	0	1
dimension	2	2	0

The $\overline{N}$ as an $\mathcal{H}$-module



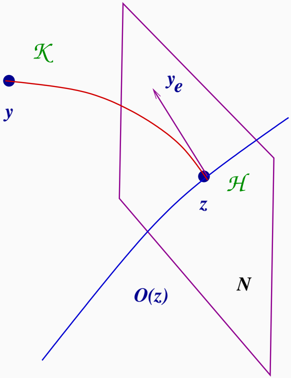
Back to geometry.

- Let $T_z(O(z)) \subseteq V$ be the tangent space of $O(z)$ at z and N be a complement.
- $T_z \subseteq V$ is an $\mathcal{H}$-module and so is $\overline{N} = V/T_z$.
- $\overline{y_e} \in \overline{N}$ and $\mathcal{H}_{\overline{y_e}}$ its stabilizer.

Theorem 1 - connecting $\mathcal{K}$ and $\mathcal{H}$

Theorem 1 (ASS)

Let $y \overset{\lambda}{\to} z$ with stabilizers Lie algebras $\mathcal{K}, \mathcal{H}$ as above. Let $\overline{N}$ be the the quotient $V/T_zO(z)$ and $\overline{y_e} \in \overline{N}$. Then we have $\hat{\mathcal{K}} \subseteq \mathcal{H}_{\overline{y_e}} \subseteq \mathcal{H}$.



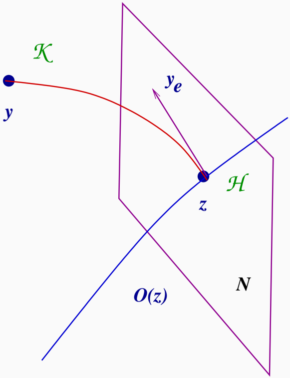
Proof: (Assume $e = d+1$). If $\mathfrak{k} \in \mathcal{K}$, then $\mathfrak{k}y = 0$. Whence $(\lambda(t)\mathfrak{k})(\lambda(t)y) = \mathfrak{k}(t)y(t) = 0$. If $\mathfrak{k}(t) = \sum_{i \geq a} t^i \mathfrak{k}_i$ and $y(t) = \sum_{j \geq d} t^j y_j$ then we have $\hat{\mathfrak{k}} = \mathfrak{k}_a$ and :

$$\begin{aligned} \hat{\mathfrak{k}} y_d = \hat{\mathfrak{k}} \hat{y} &= 0 \Rightarrow \hat{\mathfrak{k}} \in \mathcal{H} \\ \hat{\mathfrak{k}} y_e + \mathfrak{k}_{a+1} y_d &= 0 \Rightarrow \hat{\mathfrak{k}} \in \mathcal{H}_{\overline{y_e}} \end{aligned}$$

The Theorem holds even when $\gamma(t) \subset G$ is a 1-parameter family.

Theorem 1 (ASS)

Let $y \xrightarrow{\lambda} z$ with stabilizers Lie algebras $\mathcal{K}, \mathcal{H}$ as above. Let $\overline{N}$ be the the quotient $V/T_zO(z)$ and $\overline{y_e} \in \overline{N}$. Then we have $\hat{\mathcal{K}} \subseteq \mathcal{H}_{\overline{y_e}} \subseteq \mathcal{H}$.



For $\lambda(t)$ be as below, see the weight-spaces:

$$\lambda(t) = \begin{bmatrix} t^2 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathcal{G} = \left[\begin{array}{c|c|c} \mathcal{G}_0 & \mathcal{G}_{-1} & \mathcal{G}_{-2} \\ \hline \mathcal{G}_1 & \mathcal{G}_0 & \mathcal{G}_{-1} \\ \hline \mathcal{G}_2 & \mathcal{G}_1 & \mathcal{G}_0 \end{array}\right]$$

Given a $\mathfrak{k} \in \mathcal{K}_n$ with

$$\mathfrak{k} = \mathfrak{k}_{-2} + \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1 + \mathfrak{k}_2$$

The first non-zero $\mathfrak{k}_i$ determines $\hat{\mathfrak{k}}$. Thus $\hat{\mathcal{K}} = \oplus_i \hat{\mathcal{K}}_i$, with $dim(\hat{\mathcal{K}}_i) = k_i$.

Example

$X = \mathbb{C} < x_1, x_2, x_3 >$ and $G = GL_3$ acts on act on $V = Sym^4(X)$.
Let $f = (x_1^2 + x_2^2 + x_3^2)^2, g = (x_1^2 + x_2^2)^2$ and λ be as below.
Note that $f \stackrel{\lambda}{\to} g$.

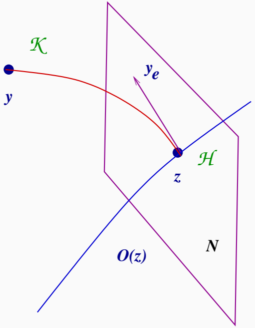
$$\mathcal{G}_f = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \; \lambda(t) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & t \end{array}\right] \; \mathcal{G}_g = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ 0 & 0 & d \end{bmatrix}$$

. $h = 2(x_1^2 + x_2^2)x_3^2$, tangent of approach.

$$\lambda(t)\mathcal{G}_f\lambda(t)^{-1} = \begin{bmatrix} 0 & a & t^{-1}b \\ -a & 0 & t^{-1}c \\ -tb & -tc & 0 \end{bmatrix} \widehat{\mathcal{G}_f} = \left[\begin{array}{cc|c} 0 & a & b \\ -a & 0 & c \\ \hline 0 & 0 & 0 \end{array}\right] \subseteq \mathcal{G}_g.$$

$\widehat{\mathcal{G}_f} = (\mathcal{G}_g)_{\overline{h}} \subset \mathcal{G}_g,\ dim((\mathcal{G}_f)_{-1}) = 2, dim((\mathcal{G}_f)_0) = 1.$

The first theorem - Geometric content



Note that...

Let $\lambda(t) = t^{\ell}$. Then $\ell' \in \mathcal{H}_0$.
How does H_0 act?
$\lambda(t)hy =$
$z + t^1(hy_1) + \ldots + t^D(hy_D)$

Theorem 1 (ASS)

In the equation
$\lambda(t)y = t^d z + t^e y_e + \ldots + t^D y_D$
$\hat{\mathcal{K}} \subseteq \mathcal{H}_{\overline{y_e}} \subseteq \mathcal{H}$ tell us that d, e, y_e are important pieces connecting z and y. y_e depends on the model!

- H_0: The interesting part of the stabilizer of z.
- $\mathcal{H}_{\overline{y_e}}/\hat{\mathcal{K}}$: The collapse of the orbit $O(y)$ as it approaches $O(z)$. Indicates simpler forms $y' \in O(y)$ with $\widehat{y'}^{\lambda} = z$.
- $\mathcal{H}/\mathcal{H}_{\overline{y_e}}$: The space of limits $z' = \widehat{y'}^{\lambda}$ obtained from elements of $y' \in O(y)$.

Therefore...

If $z = x_{nn}^{n-m} perm_m = det_n(AX_n)$, then $z = \widehat{det_n}^{\lambda}$ for a suitable 2-block λ_A. Thus $\hat{\mathcal{K}}_n \subseteq \mathcal{H}_{n,m}$. How does $\hat{\mathcal{K}}_n$ sit inside $\mathcal{H}_{n,m}$?

Recall
$X_n = \overline{X'_m} \oplus \mathbb{C}x_{nn} \oplus X_m \cong X_1 \oplus X_0$.
Then $H_{n,m}$ is as below (with $g \in H_m$):

$$\left[\begin{array}{c|c|c} * & * & * \\ \hline 0 & * & 0 \\ \hline 0 & 0 & g \end{array}\right]$$

Given a $\mathfrak{k} \in \mathcal{K}_n$ with
$\mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1$.

As per the weights of λ_A, we have:

$$\left[\begin{array}{c|cc} * & * & * \\ \hline 0 & * & 0 \\ 0 & 0 & g \end{array}\right]$$

What if $\mathfrak{k}$, $\hat{\mathfrak{k}} = \mathfrak{k}_{-1}$ for all $\mathfrak{k}$? Then the stabilizer of det_n will be tucked away from H_m! **Can λ_A be "generic"?**

Measuring Generic-ness

For $\lambda(t)$ be as below, see the weight-spaces:

$$\lambda(t) = \begin{bmatrix} t^2 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix} \; \mathcal{G} = \left[\begin{array}{c|c|c} \mathcal{G}_0 & \mathcal{G}_{-1} & \mathcal{G}_{-2} \\ \hline \mathcal{G}_1 & \mathcal{G}_0 & \mathcal{G}_{-1} \\ \hline \mathcal{G}_2 & \mathcal{G}_1 & \mathcal{G}_0 \end{array}\right]$$

Thus, for a general λ, $\hat{\mathcal{K}} = \oplus_i \hat{\mathcal{K}}_i$, with $dim(\hat{\mathcal{K}}_i) = k_i$. The vector $\overline{k} = (k_i)$ measures the generic-ness of λ vis a vis $\mathcal{K}$. The more negative the weights, the more generic is λ.

What if, λ_A is completely generic and $\overline{k}$ is as follows:

weight	-2	-1	0	1	2
dimension	$dim(\mathcal{K}_n)$	0	0	0	0

Measuring Generic-ness

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Can interesting forms and stabilizers be generic limits of det_n? Like to believe that the answer is NO

Example: det_3 - stabilizer of limit and limit of stabilizer.

Let $X = X_3$ be as below and let $det_3(X) \in Sym^3(X)$ be the usual determinant and three 2-block 1-PS with a 6-3 break:

$$\lambda_A = \begin{bmatrix} tx_1 & tx_2 & tx_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \lambda_B = \begin{bmatrix} tx_1 & x_2 & x_3 \\ x_4 & tx_5 & x_6 \\ x_7 & x_8 & tx_9 \end{bmatrix} \lambda_C = \begin{bmatrix} x_1 & tx_2 & tx_3 \\ x_4 & x_5 & tx_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$$

Let λ_D be a generic conjugate of a 3-6 break.

Now let $\lambda(t)det_3 = t^d z + \ldots +$ higher terms.

What is the limit z, stabilizer $\mathcal{H}$ and $\hat{\mathcal{K}}$ and its dimension vector?

	limit	*degree*	$dim(\mathcal{H})$	*Remark*	-1	0	1
$\hat{\mathcal{K}}_A$	det_3	1	16	$= dim(\mathcal{K}_n)$	0	16	0
$\hat{\mathcal{K}}_B$	*derangements*	0	31	$9 * 3 + 4$	12	4	0
$\hat{\mathcal{K}}_C$	$x_1 x_5 x_9$	0	56	$9 * 6 + 2$	14	2	0
$\hat{\mathcal{K}}_D$	*generic form*	0	54	$9 * 6$	16	0	0

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Alignment

A semisimple element $\mathfrak{s} \in \mathcal{K}$ is called an alignment if it commutes with λ. Consequence: $\mathfrak{s}$ stabilizes every y_i and therefore $y_d = z$.

Two questions.

- Plan A. (Alignment) Is there a common semisimple element in $g\mathcal{K}g^{-1} \cap \mathcal{H}$ while retaining that $gy \xrightarrow{\lambda} z$? What are its consequences? What if there is none?
- Plan B (Lie algebra) Are there intermediate orbits $\overline{O(z)} \subset \overline{O(w)} \subset \overline{O(y)}$ which have this property?

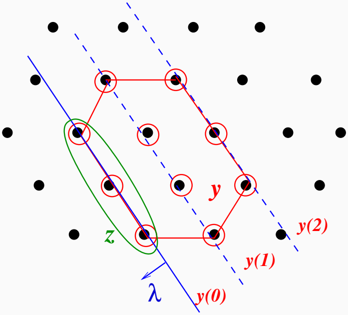
Plan B: We may have $z \xleftarrow{\lambda} w$ or $w \xleftarrow{\lambda} y' \in O(y)$ and an intermediate form to det_n and $x_{nn}^{n-m} perm_m$. What are tangent vectors $y_e \in N$ so that $\mathcal{H}_{\overline{y_e}}$ may be lifted to points $y' \in \overline{O(y)}$ such that $\widehat{\mathcal{G}_{y'}} = \mathcal{H}_{\overline{y_e}}$. –"infinitesimal determinants"

The Alignment Theorem - Weight Space

Let $T \supseteq \lambda(t)$ be a maximal torus and $\Xi(V)$, the weight space. Let $\mathcal{T} = Lie(T)$. For any $\mathfrak{t} \in \mathcal{T}$, let $t^{\mathfrak{t}}$ be the 1-PS corresponding to $\mathfrak{t}$. Let us assume that $\lambda(t)$ is such that $d = 0$, i.e.,

$$y(t) = y_0 + t^1 y_1 + \ldots + t^D y_D \text{ with } z = y_0.$$

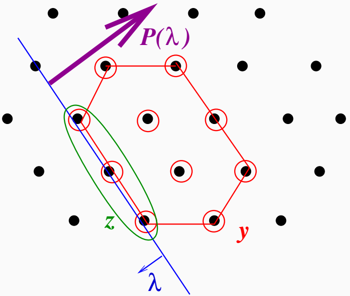
Let ℓ be such that $t^{\ell} = \lambda(t)$. Thus λ stabilizes z and $\ell \in \mathcal{H}$.



For $T \supseteq \lambda(t)$ above, let $V = \oplus_{\chi} V_{\chi}$ be the weight space decomposition. Note that $\ell \in \mathcal{T}$. We have:

$$\lambda(t)v = \sum t^{\langle \chi, \ell \rangle} v_{\chi}$$

The Alignment Theorem - The groups $P(\lambda), U(\lambda)$



Motivation

How to analyse across all g such that $gy \xrightarrow{\lambda} z$?

Recall:

- $P(\lambda) = \{p \in G | \lim_{t\to 0} \lambda(t)p\lambda(t)^{-1} \text{ exists}\}$.
- $L(\lambda)$ is precisely elements of $P(\lambda)$ which commute with λ.
- There is a Levi decomposition $P(\lambda) = L(\lambda) \ltimes U(\lambda)$, with $L(\lambda)$ reductive and $U(\lambda)$ unipotent.
- $Lie(P(\lambda)) = \mathcal{P}(\lambda) = \oplus_{i\geq 0}\mathcal{G}_i$, $Lie(U(\lambda)) = \mathcal{U}(\lambda) = \oplus_{i>0}\mathcal{G}_i$ and $Lie(L(\lambda)) = \mathcal{L}(\lambda) = \mathcal{G}_0$.

Theorem: Alignment or Nilpotency

Let $\overline{U}(\lambda) = U(\lambda(t^{-1}))$ be the *opposite* unipotent group and $\overline{\mathcal{U}}(\lambda) = \oplus_{i<0}\mathcal{G}_i$ be its Lie algebra. We then have:

$$\mathcal{G} = \overline{\mathcal{U}}(\lambda) \oplus \mathcal{L}(\lambda) \oplus \mathcal{U}(\lambda)$$

Proposition

Either there is a $\mathfrak{k} \in \mathcal{P}(\lambda) \cap \mathcal{K}$ or $\hat{\mathcal{K}} \subseteq \overline{\mathcal{U}}(\lambda)$ and is nilpotent and there is a $\mathfrak{u} \in \overline{\mathcal{U}}(\lambda)$ such that $[\mathfrak{u}, \hat{\mathcal{K}}] = 0$. For λ_A in Valiant's construction, $\mathfrak{u} \in \mathcal{H} - \hat{\mathcal{K}}$. The extra normalizer!

Theorem

Let y, z, λ be as above and $\mathcal{H} = \mathcal{G}_z$ and $\mathcal{K} = \mathcal{G}_y$. Then either (i) there is a $u \in U(\lambda)$ such that $\widehat{uy}^{\lambda} = z$ and a semisimple $\mathfrak{s} \in \mathcal{G}_{uy}$ which commutes with λ, OR (ii) $\hat{\mathcal{K}} \subseteq \mathcal{H}$ is a nilpotent Lie alegbra. There is a $\mathfrak{u} \neq \ell, \mathfrak{u} \notin \hat{\mathcal{K}}$ which normalizes $\hat{\mathcal{K}}$.

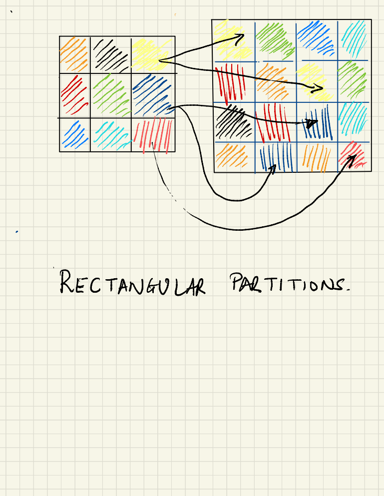
Consequences of Alignment - Rectangular Decomposition

Theorem - alignment along standard tori

If there is an alignment $\mathfrak{s} \in \mathcal{K}_n \cap \mathcal{H}_{n,m}$, the stabilizer of det_n and the padded permanent $x_{nn}^{n-m} perm_m$ via λ_A for some A. Then there is a 1-PS $u^{\mathfrak{s}} = \mu(u)$ such that the weight spaces of $X_m \dot{\cup} \{x_{nn}\}$ and X_n are linked by A.

- Variables $\{x_{11}, \ldots, x_{mm}\} \cup \{x_{nn}\}$ of $x_{nn}^{n-m} perm_m$ get partitioned into rectangles, and variables $\{x_{11}, \ldots, x_{nn}\}$ of the determinant get partitioned into rectangles.
- Each rectangle corresponds to the weight spaces w.r.t μ.
- The map A puts the permanent variables into the corresponding rectangles of the determinant.
- For both the permanent and the determinant, these rectangular spaces are also linear subspaces within their respective hypersurfaces.

Entry point for combinatorial analysis.



Alignment in Grenet's construction

- Grenet's implementation of the permanent is also via rectangular partitions

$$\left[\begin{array}{c|ccc|ccc}
0 & 0 & 0 & 0 & x_{33} & x_{32} & x_{31} \\
\hline
x_{11} & x_{77} & 0 & 0 & 0 & 0 & 0 \\
x_{12} & 0 & x_{77} & 0 & 0 & 0 & 0 \\
x_{13} & 0 & 0 & x_{77} & 0 & 0 & 0 \\
\hline
0 & x_{22} & x_{21} & 0 & x_{77} & 0 & 0 \\
0 & x_{23} & 0 & x_{21} & 0 & x_{77} & 0 \\
0 & 0 & x_{23} & x_{22} & 0 & 0 & x_{77}
\end{array}\right]$$

- $I = \{1\}\{2\}\{3\}\{7\}$ and $J = \{1,2,3\}\{7\}$ for permanent variables.
- $I = J = \{1\}\{2,3,4\}\{5,6,7\}$ for determinant variables.

- Let $\phi_A : X_m \oplus \mathbb{C} \cdot x_{nn} \to X_n$ be an embedding such that $\phi_A^*(det_n) = x_{nn}^{n-m} perm_m$. We say ϕ_A is *equivariant*, if the pull-back of $P(\lambda_A) \cap K_n$ is surjective onto H_m.
- If ϕ_A is equivariant then $n > 2^m$.
- Grenet's construction forms an important piece.

Alignment

Equivariance is complete alignment. Grenet's construction is partially equivariant. Alignment does lead to lower bounds!

Relating eigenspaces of stabilizers

The eigenspaces of semi-simple elements of $perm_n$ or $\det_n$ happen to be similar. Moreover, these are linear supspaces of the corresponding hypersurfaces.

Result (Ressayre - Mignon)

If $perm_m$ is obtained as a pull-back of det_n, then $n > m^2/2$.
Analysis of the curvature tensor of the hypersurfaces.

Proposition (ASS)

Suppose that, there is a sequence of points $(p_m) \in P_m$ and a function $k(m)$, and the guarantee that the dimension of any linear subspace $L \subseteq P_m$ containing p_m is bounded by $k(m)$. If $perm_m$ is obtained as a pull-back of $det_n(X)$ is $perm_m(W)$. Then $n \geq m^2 - k(m)$.

Conjecture: $k(m) = o(m^2)$.

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Geometric combinatorics of det_n

det_n-the master of all stabilizers

Since all forms f arise out of some det_n, perhaps all stabilizers arise out of a sequence of **good** limits:

$$det_n \stackrel{\lambda_1}{\to} F_1 \ldots \stackrel{\lambda_k}{\to} F_k = f$$

Important to analyse how $\mathcal{H}_i = \mathcal{G}_{F_i}$ change.

What is good?

- The sequence $\mathcal{K}_n = \mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_k$ reflects progression.
- Where there is alignment. Where $\mathcal{H}_i / \widehat{\mathcal{H}_{i-1}}$ is a gadget.
- For the limit $F_{i-1} \stackrel{\lambda_i}{\to} F_i$, the computability of F_i from F_{i-1} is elementary.

Geometric combinatorics of det_n

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$$det_n \xrightarrow{\lambda_1} F_1 \ldots \xrightarrow{\lambda_k} F_k = f$$

Important to analyse how $\mathcal{H}_i = \mathcal{G}_{F_i}$ change.

A key first step is F_1. Since det_n is stable with reductive stabilizer, $\overline{O(det_n)} - O(det_n)$ is of codimension 1. Suppose that:

$$\overline{O(det_n)} - O(det_n) = D_1 \cup \ldots \cup D_r \cup E_1 \cup \ldots \cup E_s$$

where $D_i = \overline{O(Q_i)}$ and $Q_i \xleftarrow{\lambda_i} det_n$, i.e., orbit closures of 1-PS limits of det_n. Let us call D_i as good divisors and E_j as bad divisors.

Plan A and B for det_n : Codimension 1 forms in $\overline{O(det_n)}$.

det_n-the master of all stabilizers

Since all forms f arise out of some det_n, perhaps all stabilizers arise out of a sequence of limits:

$$det_n \stackrel{\lambda_1}{\to} F_1 \ldots \stackrel{\lambda_k}{\to} F_k = f$$

Important to analyse how $\mathcal{H}_i = \mathcal{G}_{F_i}$ change.

Corollary (ASS)

Suppose that $W = \overline{O(Q)}$, a component of the boundary, and $Q = \widehat{det_n}^{\lambda}$. Then $\mathcal{G}_Q = \widehat{\mathcal{K}_n} \oplus \ell$ (where $t^{\ell} = \lambda$). Moreover, if there is no alignment, then $\widehat{\mathcal{K}_n}$ is nilpotent.

So must all limits Q be aligned with det_n? And what do the other divisors look like? The evidence from det_3 is Good with large subgroups of K_3 as alignments!

Alignment - The co-dimension 1 forms for det_3

Let $X = X_3$ be as below and let $det_3(X) \in Sym^3(X)$ be the usual determinant:

$$\lambda_1(t)X_3 = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & -x_1 - x_5 \end{bmatrix} + tx_9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $X = X_0 \oplus X_1$ where X_0 are trace zero matrices and $X_1 = \mathbb{C}I$, the multiples of the identity.

$$R_1 = \{X \to AXA^{-1}\} \subseteq K_3$$

Note that $R_1 \cong SL_3 \subseteq K_3$ commutes with λ_1. Then

$$\lambda_1(t)det_3 = Q_1 + tQ_1'$$

Then $\mathcal{H}_1 = \mathcal{G}_{Q_1}$ is of dimension 17, $\mathcal{H}_1 = \widehat{\mathcal{K}_3} \oplus \ell$, and $Lie(R_1) \subseteq (\mathcal{H}_1)_0$.

$$\widehat{\mathcal{K}_3} = \left[\begin{array}{c|c} * & \mathfrak{u} \\ \hline 0 & \mathfrak{r} \end{array}\right] \quad \overline{k} = \begin{array}{|c|c|c|} \hline -1 & 0 & 1 \\ \hline 8 & 8 & 0 \\ \hline \end{array}$$

8-dimensional alignment.

Let $X = X_3$ be as below and let $det_3(X) \in Sym^3(X)$ be the usual determinant:

$$\lambda_2(t)X_3 = \begin{bmatrix} 0 & -x_3 & -x_7 \\ x_3 & 0 & -x_8 \\ x_7 & x_8 & 0 \end{bmatrix} + t \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_5 & x_6 \\ x_3 & x_6 & x_9 \end{bmatrix}$$

Thus $X = X_a \oplus X_a$ where X_a is the space of anti-symmetric and X_s, symmetric matrices. Let

$$R_2 = \{X \to AXA^T | A \in SL_3\} \subseteq K_3$$

$R_2 \cong SL_3 \subseteq K_3$ commutes with λ_2.

Then

$$\lambda_2(t)det_3 = tQ_2 + t^3 Q_2'$$

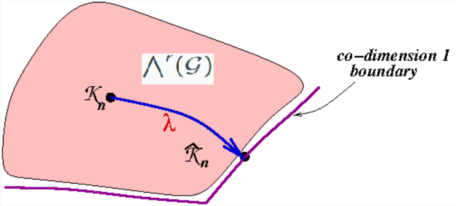
Notice top degree cancellation

Then $\mathcal{H}_2 = \mathcal{G}_{Q_2} = \widehat{\mathcal{K}_3} \oplus \ell$ and $Lie(R_2) \subseteq (\mathcal{H}_2)_0$. Structure and alignment as before.

Is this the Recipe ?

Pick a reductive $R \subset K_n$ and let $X|_R = X = \oplus_i X_i$. Choose λ suitably. Compute $\widehat{det_n}^{\lambda}$ and check cancellation.

- $W = \bigwedge^r(\mathcal{G})$ (with $r = 2n^2 - 2$) is a $G = GL(X)$-module. For any $\mathcal{L} \in W$, $G_{\mathcal{L}}$ is $N_G(\mathcal{L})$, the normalizer.
- For $\mathcal{K}_n \in W$, $G_{\mathcal{K}_n} = K_n$ is reductive and therefore $SL(X)$-stable. Its orbit isomorphic to $det_n \in Sym^n(X)$.

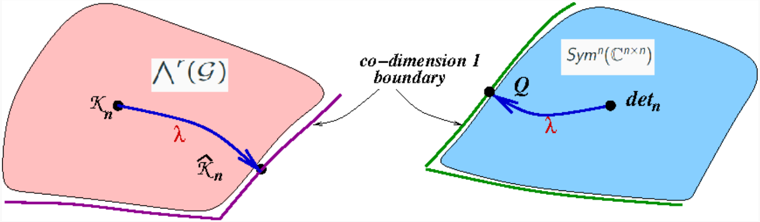


Boundary and divisors?

By Matsushima, its divisors are co-dimension 1. Again let these be $D'_1 \cup \ldots \cup D'_r \cup E$, where D'_i are obtained as limits with $D' = O(\widehat{\mathcal{K}_n}^{\lambda})$.

If λ is not aligned then $\widehat{\mathcal{K}_n}$ is nilpotent. It has an extra normalizer besides ℓ. Then $\widehat{\mathcal{K}_n}$ is *not* a divisor of $\overline{O(\mathcal{K}_n)}$.

- Let $p = (\mathcal{K}_n, det_n) \in W \times V$ and $P = \overline{O(p)} \subseteq W \times V$.



What is its boundary and divisors?

- We have $\pi_1 : P \to \overline{O(\mathcal{K}_n)}$ and $\pi_2 : P \to \overline{O(det_n)}$, surjections.
- If $Q = \widehat{det_n}^{\lambda}$ is such that $\widehat{\mathcal{K}_n}$ is nilpotent then $\widehat{\mathcal{K}_n}$ is *not* a divisor of $\overline{O(\mathcal{K}_n)}$. What is its fiber in P?

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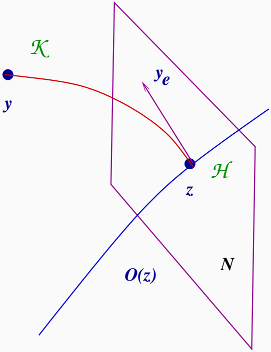
Orbit closures and stabilizer limits - Summary



- The 1-PS λ and y_e, the tangent of approach, and the containment $\hat{\mathcal{K}} \subseteq \mathcal{H}_{\overline{y_e}} \subseteq \mathcal{H}$.
- The notion of alignment - geometric as well as combinatorial entry points.
- The classical groups $P(\lambda), U(\lambda)$ and their role in the interaction with K.
- The orbit sequence for a form, the boundary of $\overline{O(det_n)}$ and that of $\overline{O(\mathcal{K}_n)}$.

For us...

The GCT approach - a relationship between stabilizers and special points. Stabilizer limits and Orbit closures illustrate the connection. Brings additional insights and classical tools to bear on the permanent vs. determinant question!

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Connecting with classical GIT limits - $G' = SL(X)$

Primary concern

Existence on invariants $\mathbb{C}[V]^{G'}$ and ability to separate orbits.
Classification of $\mathcal{N}$, i.e., of unstable $y \xrightarrow{\gamma} 0$, for a 1-PF γ.

Orbit typology:

- w is stable, if $O(w)$ is closed - $det_n, perm_m$.
- z is unstable if $0 \in O(z)$ - these form the Nullcone $\mathcal{N}$.
- y is semistable if $w \in O(y)$, $w \neq 0$, w is stable.

Hilbert, Mumford, Kempf

- 1-PS limits $y \xrightarrow{\lambda} 0$ detect closure.
- Existence of an optimal μ and a canonical parabolic subgroup $P_y = P(\mu)$, with $G_y \subseteq P_y$.
- Generalized to $y \xrightarrow{\gamma} w$, where $S = O(w)$ is closed, i.e. w is stable and y is semistable.

For Kempf-optimal μ and Luna

For Kempf-optimal μ, the classical situation presents two cases:

$$\mu(t)y = t^d z + t^e y_e \ldots + t^D y_d \text{ with } d \geq 0$$

- $z = 0$ and $y \in \mathcal{N}$ (such as the padded $x_{nn}^{n-m} perm_m$).
  - μ can be chosen to align with any reductive subgroup of $K = G_y$. In fact, $K \subseteq L(\mu) \subseteq P(\mu) = P_y$.
  - y_e, tangent of approach is unique up to $U(\mu)$.
  - If $\hat{\mathcal{K}} \subsetneq \mathcal{G}_{y_e}$, then $\overline{O(y_e)}$ is the intermediate orbit.
- $d = 0$ and z is stable (such as $perm_m, det_n$) and y is semi-stable.
  - The stabilizer H, of z is reductive and there is an H-module N complement to the orbit.
  - We may choose $y' \in N \cap O(y)$, and $\lambda \subseteq H$. (Luna) $G \times^H N$ is a local model of the vicinity of $O(z)$. We are in Case 1.

Thus in both cases, alignment holds and intermediate tangent varieties exist.

Moreover...

- For every unstable z, and Kempf-optimal μ, we have:

$$\mu(t)z = t^d\overline{z} + \text{ higher terms}$$

  This $\overline{z}$ is the tangent of the optimal path which takes z to 0.
- Let $\overline{\mathcal{N}} = \{\overline{z} | z \in \mathcal{N}\} \subseteq T_0\mathcal{N}$ is related to the Hesselink strata of $\mathcal{N}$ and their $P(\mu)$-structure.

What happens when $y \xrightarrow{\lambda} z$ (with y_e as the tangent) and z is partially stable ($x_{nn}^{n-m} perm_m$) and y is stable (det_m)?

- Then (i) $\lambda \in P_z = P(\mu)$ and (ii) μ may be chosen such that $y \xrightarrow{\lambda} z \xrightarrow{\mu} \overline{z}$ and λ and μ commute. For $z = x_{nn}^{n-m} perm_m$, we have $z = \overline{z}$.
- $\overline{\mathcal{N}}(z) = \{\overline{w} \in \overline{N} | w$ is a tangent for some $y \xrightarrow{\lambda} z\}$.
- $K \not\subset P(\mu)$ but $K \subset \cup_{w \in W} P(\mu) w P(\mu)$ for some collection W.

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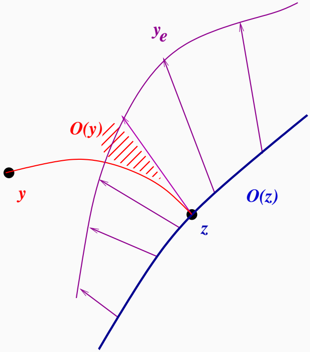
Plan B - More Pictures - The tangent vector

Lets look at..

The two block case and $\hat{\mathcal{H}} \subseteq \mathcal{H}_{\overline{y_e}} \subseteq \mathcal{H}$.



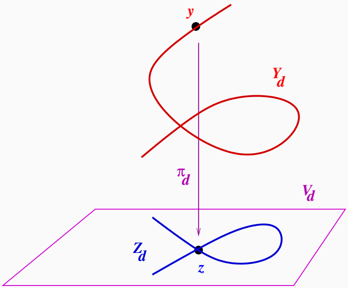
This examines the gap $\hat{\mathcal{K}} \subsetneq \mathcal{H}_{\overline{y_e}}$. Then $dim(O(y))$ in V is greater than $dim(O(\overline{y_e}))$ in $G \times^H \overline{N}$.

So is there...

an element $w \in V$ with stabilizer $\mathcal{H}'$ such that $\widehat{\mathcal{H}'}^{\mu} = \mathcal{H}_{\overline{y_e}}$? Is there an "extension" of y_e into V?

Would indicate $\overline{O(z)} \subsetneq \overline{O(w)} \subsetneq \overline{O(y)}$, help in finding forms simpler than det_n with $x_{nn}^{n-m} perm_m$ as limits.

Plan B - More Pictures - Co-limits



This examines the gap $\mathcal{H}_{\overline{y_e}} \subsetneq \mathcal{H}$.
Let $Y_d = O(y) \cap V_{\geq d}$ and $Z_d = \pi_d(Y_0)$. Note that $y \in Y_d$ and $z \in Z_d$, the space of co-limits of z. Let $Z = \overline{O(Z_d)}$, then $\overline{O(z)} \subseteq Z \subseteq \overline{O(y)}$ is an intermediate variety.

What is $T_z Z_d$?

Let $\mathcal{G}_{y,d} = \{\mathfrak{g} \in \mathcal{G} | \mathfrak{g}y \in V_{i \geq d}\}$. Then $\pi_d(\mathfrak{g} \cdot y) = T_z Z_d$. How does H_0 act?

The Claims B1 and B2

- If $dim(\mathcal{H}_{\overline{y_e}}/\hat{\mathcal{K}}) > 0$ then there is a suitable extension of y_e into V.
- $dim(\mathcal{H}/\mathcal{H}_{\overline{y_e}})_{(-1)} > 0$ indicates the presence of a $z' \notin O(z)$.

Others forms in $\overline{O(det_n)}$

Let $X_m \subset X_n$ as before. Let $A_1, A_2 : X_m \to X_m$ be two linear maps and let B_1, B_2 be the $m \times m$-matrices $B_i = A_i X_m$, i.e., with entries as formal linear combinations of entries of X_m. Let $f_i = det(B_i)$, then $f_i \in \overline{O(det_m)}$. Let G be the $r \times r$-gadget matrix constructed out of B_1 and B_2 such that $det(G) = f_1 + f_2$. Let Y be the $n \times n$-matrix below:

$$\begin{bmatrix} G & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

Then $f = det(Y) = f_1 + f_2 \in Sym^m(X_m)$, is of degree m. The homogenization of f is indeed $f' = x_{nn}^{n-m} f \in Sym^n(X_n)$, and thus $W = \overline{O(f')} \subseteq \overline{O(det_n)}$ and we have the surjection.

$$\mathbb{C}[\overline{O(det_n)}] \twoheadrightarrow \mathbb{C}[W]$$

What are the G-modules in $\mathbb{C}[W]$?