Computational complexity of representation theoretic multiplicities and characters

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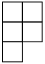
Partitions and tabaleaux

Integer partitions and Young diagrams:

$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, $\lambda_1 + \lambda_2 + \cdots = n$.

$\lambda = (2, 2, 1)$



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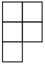

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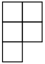

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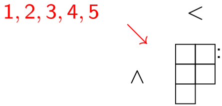

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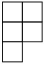

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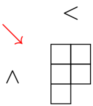
$1, 2, 3, 4, 5$



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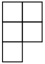

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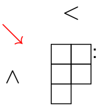

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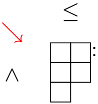

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Representations of S_n

Symmetric group S_n – permutations under composition:

$$\pi : [1, 2, \ldots, n] \xrightarrow{\sim} [1, 2, \ldots, n], \qquad \pi\sigma = \pi(\sigma)$$

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Example: if $V = \mathbb{C}^3$, $\pi \in S_3$, set $\pi(e_i) := e_{\pi_i}$ for $i = 1..3$, so e.g. $231 \to \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

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The **irreducible representations** of S_n: the *Specht modules* $\mathbb{S}_\lambda$

$$V = \underbrace{\mathbb{C}\langle e_1 + e_2 + e_3 \rangle}_{\mathbb{S}_{(3)}} \oplus \underbrace{\mathbb{C}\langle e_1 - e_2, e_2 - e_3 \rangle}_{\mathbb{S}_{(2,1)}}$$

Basis indexed by SYTs of shape λ, so $\dim \mathbb{S}_\lambda = f^\lambda := \#\{T : \text{ SYT, shape } \lambda\}$.

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Characters: $\chi^\lambda(\alpha) = \chi^\lambda(\pi) := \mathit{Trace}\ \rho^\lambda(\pi)$, for π of cycle type α.

$$\underbrace{\chi^V(\pi = 231)}_{=0} = \underbrace{\chi^{(3)}(\pi)}_{=1} + \underbrace{\chi^{(2,1)}(\pi)}_{=-1}$$

Representations of the General Linear group $GL_N(\mathbb{C})$

Irreducible (polynomial) representations of $GL_N(\mathbb{C})$:
Weyl modules V_λ, indexed by highest weights λ, $\ell(\lambda) \leq N$.
Basis indexed by **Semi-Standard Young tableaux** of shape λ :

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Characters: Schur functions

$$s_\lambda(x_1, \ldots, x_N) = \sum_{T \in SSYT(\lambda)} x^{type(T)}$$

$$s_{(2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$$

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Theorem (Schur-Weyl duality)

Under the joint action of the groups S_n and $GL(V)$, the tensor space decomposes as:

$$V \otimes V \otimes \cdots \otimes V = \sum_{\lambda \vdash n} \mathbb{S}^\lambda \otimes V_\lambda.$$

Structure constants (multiplicities) I

Tensor product of irreducible GL representations:

$$V_\lambda \otimes V_\mu = \oplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

Littlewood-Richardson coefficients: $c^\nu_{\lambda\mu}$

$$V_{(2,1)} \otimes V_{(2,1)} = V_{(4,2)} \oplus V_{(4,1,1)} \oplus V_{(3,3)} \oplus V_{(3,2,1)}^{\oplus 2} \oplus V_{(3,1,1,1)} \oplus V_{(2,2,2)} \oplus V_{(2,2,1,1)}$$

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Theorem (Littlewood-Richardson, stated 1934, proven 1970's)

The coefficient $c^\nu_{\lambda\mu}$ is equal to the number of LR tableaux of shape ν/μ and type λ.

			1	1	1
	1	2	2		
2	3	3			

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(LR tableaux of shape $(6,4,3)/(3,1)$ and type $(4,3,2)$. $c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

Structure constants (multiplicities) II

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

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Plethysm coefficients: $GL_n \xrightarrow{\rho_\nu} GL_m \xrightarrow{\rho_\mu} GL_N$: $\rho_\mu \circ \rho_\nu : GL_n \to GL_N$:

$$\rho_\mu(\rho_\nu) = \bigoplus_\lambda V_\lambda^{\oplus a_\lambda(\mu[\nu])}$$

$a_\lambda(d[n])$ – multiplicity of V_λ in $Sym^d(Sym^n V)$ under GL action.

$$\rho_{(2)}[\rho_{(2)}] \simeq V_{(4)} \oplus V_{(2,2)}$$

Major problems in Algebraic Combinatorics

[Murnaghan, 1938]: $c^{\lambda}_{\mu\nu} = g\left((N-|\lambda|,\lambda),(N-|\mu|,\mu),(N-|\nu|,\nu)\right)$ for $|\lambda| = |\mu| + |\nu|$ and N-large.

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Find a positive combinatorial interpretation for $g(\lambda,\mu,\nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda,\mu,\nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$.

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Combinatorial formulas for $g(\lambda,\mu,\nu)$:

- Two two-row partitions [*Remmel–Whitehead*, 1994; *Blasiak–Mulmuley–Sohoni*,2015] ;
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- One hook $\nu = (n-k,1^k)$ [*Blasiak 2012, Blasiak-Liu 2014*]
- Other special cases [Bessenrodt-Bowman, Colmenarejo-Rosas, Garsia, Goupil, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Tewari, Vallejo, Chenchen Zhao].

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Applications beyond Combinatorics: **Geometric Complexity Theory** (VP vs VNP), **Quantum Information Theory** (quantum marginal problem) etc

Geometric Complexity Theory in a Nutshell

VP vs VNP: determinant vs permanent

$$\mathrm{det}_n := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)} \qquad \mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} x_{i,\sigma(i)}$$

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$$\bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \delta_{\lambda,d,n}} \simeq \mathbb{C}[\overline{GL_{n^2}\det_n}]_d \overset{?}{\twoheadrightarrow} \mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]_d \simeq \bigoplus_{\lambda \vdash nd} V_\lambda^{\oplus \gamma_{\lambda,d,n,m}},$$

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Geometric Complexity Theory in a Nutshell

VP vs VNP: determinant vs permanent

$$\det_n := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)} \qquad \mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} x_{i,\sigma(i)}$$

Conjecture [Valiant'78, VP $\neq$ VNP]:
The (normalized) permanent $x_{11}^{n-m}\mathrm{per}_m \neq \det_n[Ax^T]$ for $n = poly(m)$.

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Theorem (Bürgisser-Ikenmeyer-P)

There are no such occurrence obstructions for $n > m^{25}$.

Kronecker coefficients and GCT

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$$\delta_{\lambda,d,n} \leq g(\lambda, n^d, n^d) \qquad \gamma_{\lambda,d,n,m} \leq a_\lambda(d[n])$$

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There exist λ*, s.t.* $g(\lambda, n^d, n^d) = 0$ *(so* $mult_\lambda \mathbb{C}[GL_{n^2}\det_n] = 0$*) and* $\gamma_{\lambda,d,n,m} > 0$ *for some* $n > poly(m)$*.*

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Theorem (Ikenmeyer-P)

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n^d, n^d) = 0$, then $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}\mathrm{per}_m^n}]) = 0$.

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If $\ell(\lambda) \leq m^2$, $\lambda_1 \geq nd - md$, $d > 3m^3$, and $n > 3m^4$, then $g(\lambda, n \times d, n \times d) > 0$, except for 6 special cases.

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Theorem (Ikenmeyer-P)

For every partition ρ, let $n \geq |\rho|$, $d \geq 2$, $\lambda := (nd - |\rho|, \rho)$. Then $g(\lambda, n^d, n^d) \geq a_\lambda(d[n])$.

Complexity of Computing Multiplicities I

Littlewood-Richardson coefficients: $c^{\lambda}_{\mu\nu} = \mathrm{mult}_{\lambda} V_{\mu} \otimes V_{\nu} = \#LR - tableaux$

$c^{(6,4,3)}_{(4,3,2)(3,1)} = 2$:

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Conjecture (Pak-Panova)

ComputeLR is strongly #P-complete, i.e. when input is in unary (input size is $O(n)$*).*

(Related to counting 2d contingency tables, and graphs with given degree sequence)

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`Input:` λ, μ, ν
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PlethPos:
`Input:` λ, d, n
`Output:` Is $a_\lambda(d[n]) > 0$?

ComputePleth:
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`Output:` Value of $a_\lambda(d[n])$.

Fischer-Ikenmeyer: PlethPos is NP-hard, ComputePleth is #P-hard.

Quantum algorithms for Kronecker coefficients

Theorem (Bravyi-Chowdhury-Gosset-Havlicek-Zhu'23)

KronPos in in QMA. The problem of computing $f^\lambda f^\mu f^\nu g(\lambda,\mu,\nu)$ is in #BQP.

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Let $\lambda, \mu, \nu \vdash n$ and k be a constant, such that $f^\nu \leq n^k$. Then $g(\lambda, \mu, \nu)$ can be computed in time $O(n^{4k^2+1})$.

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Proof sketch: Asymptotics: If $f^\nu \le n^k$, then $n - \nu_1 \le 4k^2$.

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1. $O(n^{d\ell})$ *where* $\ell = \ell(\lambda)$.
2. $O(n^{4K^3(K+1)})$ *where* $f^\lambda \leq n^k$ *and* $K = 4k^2$ *for arbitrary* d, m.

In particular, we have a polynomial time algorithm for computing $a^\lambda_{d,m}$ if either d and $\ell(\lambda)$ are fixed, or d grows but the dimension f^λ grows at most polynomially.

[Kahle-Michalek'15]: Poly-time algorithm when d, ℓ-fixed.
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Proof sketch: counting points in polytopes Q:

$$a^\lambda_{d,m} = \sum_{\sigma \in S_{K+1}} \mathrm{sgn}(\sigma) \sum_{r=1}^{4K^3+1} \sum_{(c_1,\ldots,c_{r-1}) \in [1,2K]^{r-1}} \sum_{\bar{j} \in [K+1]^{r-2}} |Q(\bar{j}, c, \hat{\lambda} + \delta(K) - \sigma(\delta))|$$

Characters of S_n

characters: $\text{char}\,\mathbb{S}_\lambda = \chi^\lambda : S_n \to \mathbb{C}$

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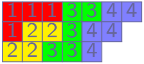
Murnaghan–Nakayama rule:

$$\chi^\lambda[\alpha] = \sum_{T \,:\, \text{MN tableaux, shape } \lambda, \text{ content } \alpha} (-1)^{ht(T)}$$



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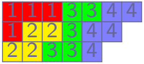

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Key players:

$$g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda[w]\chi^\mu[w]\chi^\nu[w].$$

Characters of S_n

	id	$(1,2)$	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4)$
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Theorem (Ikenmeyer-Pak-P'22)

ComputeCharSq $\notin$ #P *unless* $PH = \Sigma_2^{\mathsf{P}}$.

No nice combinatorial interpretation for $\chi^\lambda(\alpha)^2$

Set partitions

Ordered set partitions of items $\mathbf{a} = (a_1, \ldots, a_m)$ into bins of sizes $\mathbf{b} = (b_1, \ldots, b_k)$:

$$P(\mathbf{a}, \mathbf{b}) := \#\{(B_1, B_2, \ldots, B_k) : B_1 \sqcup B_2 \sqcup \ldots \sqcup B_k = [m], \sum_{i \in B_j} a_i = b_j \text{ for all } j = 1, \ldots, k\}$$

$$P((1,1,1,1,1,2,2,3),(\mathbf{4},\mathbf{4},\mathbf{4})) = |\{(\underbrace{1+1+2}_{4}, \underbrace{1+3}_{4}, \underbrace{1+1+2}_{4}), \ldots\}| = 245$$

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Proposition (IPP)

Let $\mathbf{c}$ *and* $\mathbf{d}$ *be two sequences of nonnegative integers, such that* $|\mathbf{c}| = |\mathbf{d}| + 6$. *Then there are partitions* λ *and* α *of size* $O(\ell|\mathbf{c}|)$ *determined in linear time, such that*

$$\chi^\lambda(\alpha) = P(\mathbf{c}, \overline{\mathbf{d}}) - P(\mathbf{c}, \overline{\mathbf{d}'}),$$

where $\overline{\mathbf{d}} := (2, 4, d_1, d_2, \ldots)$ *and* $\overline{\mathbf{d}'} := (1, 5, d_1, d_2, \ldots)$.

3- and 4d Matchings

Proposition (IPP)

For $\forall$ *two independent 3d matching problem instances* E *and* E', $\exists$ $\mathbf{c}$ *and* $\mathbf{d}$, *such that*

$$\#3DM(E) - \#3DM(E') = \frac{1}{\delta}\left(P(\mathbf{c},\overline{\mathbf{d}}) \; - \; P(\mathbf{c},\overline{\mathbf{d'}})\right) = \frac{1}{\delta}\chi^{\lambda}(\alpha).$$

where δ *is a fixed multiplicity factor, number of orderings.*

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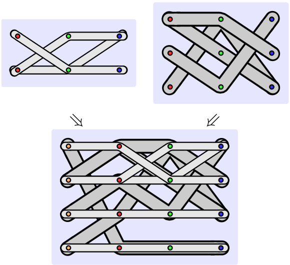

Vertices $[4] \times [4]$ and hyperedges $J =$
$(1,1,2,2),(2,2,1,1),(2,2,2,1),(3,3,3,3),(4,4,4,4),$
$(2,1,1,2),(2,1,2,3),(3,2,3,1),(4,3,1,3),(1,4,4,4)$

$\rightarrow$ encoded via vectors $[v_1,\ldots,v_{10}]$
$\rightarrow$ items of size $v_1 + v_2 r + \cdots + v_{10} r^9$

Vertix encodings:
$\{[0^{j-1},1,0^4,i,0^{4-j},3] \mid i \in [4], j \in [4]\}$
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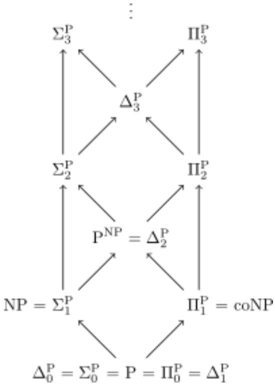
Hyperedge $(1,1,2,2)$
$\rightarrow [0^4,1,4-1,4-1,4-2,4-2,0]$
Bins size $b_1 = [1^5,4^4,12]$, bins: $\mathbf{b} = (b_1^{10})$:

$$[0,0,0,0,1,3,3,2,2,0]+[1,0,0,0,0,1,0,0,0,3]+[0,1,0,0,0,0,1,0,0,3]$$
$$+[0,0,1,0,0,0,0,2,0,3]+[0,0,0,1,0,0,0,0,2,3]=[1,1,1,1,1,4,4,4,4,12]$$

Characters are as hard as the polynomial hierarchy

Theorem (Ikenmeyer-Pak-P'22)

Let $\chi^2 : (\lambda, \pi) \mapsto \left(\chi^\lambda(\pi)\right)^2$, *where* $\lambda \vdash n$ *and* $\pi \in S_n$. *If* $\chi^2 \in \#\mathrm{P}$, *then the polynomial hierarchy collapses to the second level:* $\mathrm{PH} = \Sigma_2^{\mathrm{p}} = \mathrm{NP}$ [1].



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[1] A hypothesis widely believed to be false, similar to $\mathrm{P} \neq \mathrm{NP}$

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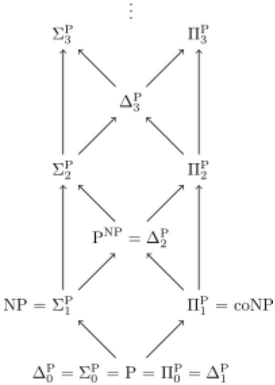

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$$\Longrightarrow [\chi = 0] \text{ is } \mathsf{C_{=}P} := [\underbrace{\mathsf{GapP}}_{\#\mathsf{P}-\#\mathsf{P}} = 0]\text{-complete.}$$

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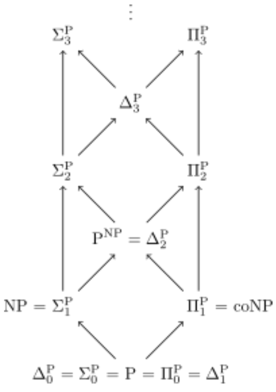

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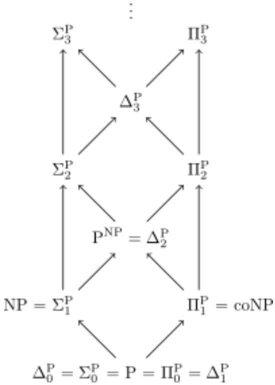

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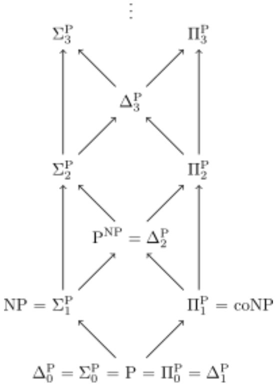

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The End

Computing Kronecker, plethysm coefficients and especially S_n characters...

Vielen Dank für Ihre Aufmerksamkeit!