Asymptotic tensor rank is characterized by polynomials

Harold Nieuwboer

Joint work with
Matthias Christandl, Koen Hoeberechts, Péter Vrana, Jeroen Zuiddam

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QMATH

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The **tensor rank** $\mathrm{R}(T)$ of T is the smallest r such that

$$T = \sum_{i=1}^{r} u_{1,i} \otimes \cdots \otimes u_{k,i}$$

for some vectors $u_{j,i} \in \mathbb{F}^{d_j}$.

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$$\begin{aligned} W &= e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \\ &= |001\rangle + |010\rangle + |100\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \end{aligned}$$

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  Rank of $\mathrm{MM}_n = \mathrm{MM}_{n,n,n}$: ?

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Asymptotic tensor rank of a tensor T:

$$\underset{\sim}{\mathrm{R}}(T) = \lim_{n\to\infty} \mathrm{R}(T^{\boxtimes n})^{\frac{1}{n}}$$

Well-defined: $\mathrm{R}(T_1 \boxtimes T_2) \leq \mathrm{R}(T_1)\mathrm{R}(T_2)$, and Fekete's lemma.

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Best bound: $2 \leq \omega \leq 2.371339$ [2]

Our results

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Theorem: Let $r \in \mathbb{R}$, $d_1, \ldots, d_k \in \mathbb{N}$, $V = \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$. Then

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is Zariski-closed, i.e., there exist polynomials $f_1, \ldots, f_\ell$ such that $V_{\leq r}$ is the joint zero set of $f_1, \ldots, f_\ell$:

$$V_{\leq r} = \{T \in V : f_i(T) = 0 \; \forall i = 1, \ldots, \ell\}$$

Zariski-closedness

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Follows easily from main result. Other direction: Need to show that $V_{\leq r}$ is Zariski-closed. Now

$$\underset{\sim}{\mathrm{R}}[\overline{V_{\leq r}}] = \underset{\sim}{\mathrm{R}}[V_{\leq r}] \leq r$$

so every $T \in \overline{V_{\leq r}}$ has $\underset{\sim}{\mathrm{R}}(T) \leq r$, hence $T \in V_{\leq r}$, so $\overline{V_{\leq r}} \subseteq V_{\leq r}$.

Computability

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Corollary: Let $\mathbb{F}$ be a "computable field".
For every $r \in \mathbb{R}$, **there exists** an algorithm which determines for given $d_1, \ldots, d_k$ and $T \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$, whether $\underset{\sim}{\mathrm{R}}(T) \leq r$.

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Corollary: The set $\mathcal{R} = \{\underset{\sim}{\mathrm{R}}(T) : T \in \mathbb{F}^{d_1} \otimes \ldots \otimes \mathbb{F}^{d_k}\}$ is well-ordered: any sequence $r_1 \geq r_2 \geq \ldots$ in $\mathcal{R}$ eventually stabilizes.

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$\mathcal{R} = \cup_d \mathcal{R}_d$, and new asymptotic tensor ranks grow with d. ∎

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"Weak" form of the conjecture: $V_{\leq r}$ are always **irreducible**. Dimension argument gives a bound on the number of different ranks that can appear in $\mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$.

Proof of the main result

A small lemma

Fix $V = \mathbb{F}^{d_1} \otimes \ldots \otimes \mathbb{F}^{d_k}$. For $A \subseteq V$, write $A^{(n)} = \{T^{\otimes n} : T \in A\}$.

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$$\operatorname{span} A^{(n)} = \bigcap \ker l$$

over linear forms $l : \operatorname{Sym}^n(V) \to \mathbb{F}$ which vanish on $A^{(n)}$.

If l is a linear form on $\operatorname{Sym}^n(V)$ vanishing on $A^{(n)}$, then $f(T) = l(T^{\otimes n})$ is a polynomial vanishing on A.

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Proof of the main result

Let $T \in \overline{A}$. For $n \in \mathbb{N}$, there are linearly independent $S_1, ..., S_{p(n)} \in A^{(n)}$ and $\alpha_1, ..., \alpha_{p(n)} \in \mathbb{F}$ s.t. $T^{\otimes n} = \sum_j \alpha_j S_j^{\otimes n}$.

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Tensor rank is subadditive:

$$\mathrm{R}(T^{\otimes nm}) \leq p(n)^m \max_{i_1,\ldots,i_m \in [p(n)]} \mathrm{R}\left(\bigotimes_{j=1}^{m} S_{i_j}^{\otimes n} \right).$$

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There exist m_i summing to m such that

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$m \to \infty$ gives $\underset{\sim}{\mathrm{R}}(T) \leq p(n)^{\frac{1}{n}}(\underset{\sim}{\mathrm{R}}[A] + \varepsilon)$. Then $\varepsilon \to 0$ and $n \to \infty$ yields $\underset{\sim}{\mathrm{R}}(T) \leq \underset{\sim}{\mathrm{R}}[A]$. ■

Extension to other parameters

1. Zariski-closedness + computability extends to regularizations of **admissible** functionals on a vector space:
   - subadditivity
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2. Includes points in Strassen's asymptotic spectrum, such as the quantum functionals.
3. For spectral points, well-orderedness also holds across different formats using a new **growth** argument for higher-order tensors. Implies new lower bounds on the asymptotic subrank of k-tensors for $k > 3$.

Bonus

Closedness of $\mathcal{R}$ over $\mathbb{C}$

Theorem: Over $\mathbb{C}$, $\mathcal{R}$ is closed: any limit of a sequence of asymptotic ranks, is itself an asymptotic rank.

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If r is not itself an asymptotic rank,

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This is impossible by the Baire property for algebraic varieties. ■

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Thank you!

Appendix

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