Trading Determinism for Noncommutativity in SINGULARITY Testing

Joint work with V. Arvind and Abhranil Chatterjee

Edmonds' Problem (1967)

$X = \{x_1, \ldots x_n\}$ $\quad \mathbb{F}$: field

$T = A_0 + A_1 x_1 + \cdots + A_n x_n$

$A_i \in M_s(\mathbb{F})$

Problem: Check if T is invertible over $\mathbb{F}(x_1, \ldots x_n)$ $\longrightarrow$ SINGULAR

Motivation: Matching, Matroids (Lovász)

- Randomized poly-time

- [Main Open Q.] Deterministic Algorithm?

  $\hookrightarrow$ Ckt Lower Bound (KI'04)

NSINGULAR PROBLEM

- $X = \{x_1, \ldots x_n\} \rightarrow$ Noncommuting variables
- Rank in Free Skew Field $\rightarrow \mathbb{F}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$

$$\mathbb{F}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$$
$$\uparrow$$
$$\mathbb{F}\langle x_1, \ldots, x_n \rangle$$

[Cohn, Amitsur]

C-Rank / NC-Rank

$$M = \begin{bmatrix} 0 & 1 & x_1 \\ -1 & 0 & x_2 \\ -x_1 & -x_2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & x_2x_1 - x_1x_2 \end{bmatrix}$$

$\Longrightarrow$ C-rank $= 2$, NC-rank $= 3$,

Known Theorem

Thm [GGOW'16, IQS'18, HH'21]:

NSINGULAR $\in$ P.

- $\rightarrow$ No direct connection with traditional PIT methods.

Key Concepts

Inner Rank:

$$\text{nc-rank}(T) = \min r \text{ s.t}$$

$$T = P_{s \times r} Q_{s \times r}$$

Blow-up spaces

$$T = A_0 + \sum_{i=1}^{n} A_i x_i$$

$$\underline{p} = (p_1, p_2, \dots, p_n) \in M_d(\mathbb{F})^n$$

$$T(\underline{p}) = A_0 \otimes I_d + \sum_{i=1}^{n} A_i \otimes M_i$$

$$T^{\{d\}} = \{ T(\underline{p}) : \underline{p} \in M_d(\mathbb{F})^n \}$$

Key Fact [DM, IQS]

- [Regularity Lemma]: The maximum rank in $T^{\{d\}}$ is a multiple of d.

- Def^n: $\text{nc-rank}(T) = \lim_{d \to \infty} \frac{\text{rank}(T^{\{d\}})}{d}$.

  $= \text{Inner Rank}$.

Concept of Witness:

$\underline{p} = (p_1, \ldots, p_n) \in M_d^n(\mathbb{F})$ is witness of $\text{ncrank}(T) \geq r$ if $\text{rank}(T(\underline{p})) \geq rd$.

↳ Key idea: Find better and better witness.

[Sketch] : NSINGULAR $\in$ P.

Input : $T = A_0 + \sum A_i x_i$

Output : ncrank (T).

→ Suppose already have witness p of rank $= r$ in d-dimension.

1. Is r the maximum rank ?

   If "YES" → NC-Rank $= r$ [STOP].

2. Otherwise : Use a "rank - increment" step to produce witness of rank $> r$

3. [Rounding] : Another witness of rank $\geq r+1$

4. [Blow-up Control] : Keep the dimension small.

IQS : [Finite dimensional division algebra, Simple linear algebra]

> A new insight.

nc-PIT - based "rank increment"

- Let $(p_1, \ldots, p_n)$ be a rank witness p of dimension $= d$

Shift:

$z_1, \ldots z_n$: generic matrices

$$T_d(z_1 + p_1, \ldots, z_n + p_n)$$

$$\rightarrow U \begin{bmatrix} I_{rd} - L & 0 \\ 0 & C - B(I_{rd} - L)^{-1} A \end{bmatrix} V$$

→ Conceptually similar: [BBJP'19]

Key Observation

- rank can be increased

$$\Leftrightarrow \quad C - B(I_{rd} - L)^{-1} A \neq 0$$

$$\Leftrightarrow \quad C - B\left(\sum_{k \leq rd} L^k\right) A \neq 0$$

[Schützenberger]

- Final step: $\rightarrow$ nc ABP PIT
  [Raz-Shpilka'05]

Steps:

Rank increment

$\downarrow$

Constructive Regularity Lemma

$\downarrow$

Blow-up Control

Trade-off Result

$$X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_k$$

X_i : Noncommuting

$i \neq j$: $[X_i, X_j] = 0$

Main Question:

- Given a linear matrix T over X, compute the rank of T.

$\mathcal{U}_{[k]}$: [KVV'20]

$\uparrow$

$\mathbb{F}\langle x_1 \sqcup x_2 \sqcup \cdots \sqcup x_n \rangle$

Main Theorem 1

A polynomial-time algorithm for the [PC]-SING problem for $K = O(1)$, $\mathbb{F} = \mathbb{Q}$.

Algebraic Question: ← A central problem in algebraic automata theory

- Let $[T_1]_{8\times 8}, [T_2]_{8\times 8}$ are linear matrices defined over $X = X_1 \sqcup \cdots \sqcup X_K$, $K = O(1)$, u_1, v_1, u_2, v_2 : vectors.

Check: $\vec{u}_1 \left(\sum_{i \geq 0} T_1^{\,i} \right) \vec{v}_1 \overset{?}{=} \vec{u}_2 \left(\sum_{i \geq 0} T_2^{\,i} \right) \vec{v}_2$

Solution: In deterministic polynomial-time.

Example: $X_1 = \{x_1, x_2\}$. $X_2 = \{x_1', x_2'\}$

$\Rightarrow x_1 x_2' x_2 x_1' \sim x_1 x_2 x_2' x_1'$

History

- Rabin & Scott - 1959
- Griffiths - 1968
- Bird - 1973
- Valiant - 1974
- Friedman-Greibach - 1982
- Harju - Karhumäki - 1991
- Worrell - 2013 ← Randomized Poly-time

A Finite Truncation [Worrell, Schützenberger]

$\Downarrow$

An ABP identity testing problem over X

(Glimpse of Proof Idea)

- Main point: The new NSINGULAR can be lifted in this setting.
  - ↳ Do not know how to use other algorithms!

↳ particularly PIT connection

Two Recursive Subroutines

- PC-PIT[k]
  - ↓ Finds nonzero for a PC-ABP
- PC-RANK[k]
  - ↓ Finds rank witness matrices.

Sketch Contd.

$$X_1 \longrightarrow M_1 \otimes I_{d_2} \otimes I_{d_3} \otimes \cdots \otimes I_{d_k}$$

$$X_2 \longrightarrow I_{d_1} \otimes M_2 \otimes I_{d_3} \otimes \cdots \otimes I_{d_k}$$

$$\vdots$$

$$X_k \longrightarrow I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes M_k$$

partially - commutative structure.

PC-RANK (k)

$\Downarrow$

PC-PIT(k) $\leftarrow$ very high level

$\Downarrow$

PC-RANK(k-1)

– Rounding and Blow-up Control

→ Somewhat sophisticated than IQS'18

Some Remarks

• Run time — $s^{2^{O(k \log k)}}$

↳ Worrel : Randomized $s^{O(k)}$

Improve Run time.

• For non constant k : No hardness result is known (for the series equivalence problem)

• No invariant theory connection known. [MW'20]

A central Open Question:

- Design efficient Black-box algorithm for NSINGULAR

Quasi-NC algorithm for Bipartite PM

[FGT'21]

More Sketch

PC- PIT[k] $\rightsquigarrow$ Output: Assignment for non-zero.

$\downarrow$

Homogenize (w.r.t X_1)

$\downarrow$

Implement [RS'05] over a more complicated domain

$\Downarrow$

Reduction to PC-RANK[k-1]

$\Downarrow$

PC- PIT[k-1].

Intertwine

PC- RANK [k]

$\hookrightarrow$ Output: $I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes M_x \otimes \cdots \otimes I_{d_n}$

$\Downarrow$

PC- PIT [k] $\rightarrow$ Used in Rank increment step.

Blow-Up Control and Rounding steps:

$\hookrightarrow$ Conceptually [IQS]

More work to implement.

Thank You!