

Trading Determinism for Noncommutativity in SINGULARITY Testing

Joint Work with V. Arvind and Abhranil Chatterjee

Edmonds' Problem (1967)

$$X = \{x_1, \dots, x_n\} \quad \mathbb{F}: \text{field}$$

$$T = A_0 + A_1 x_1 + \dots + A_n x_n$$

$$A_i \in M_b(\mathbb{F})$$

Problem: Check if T is invertible
over $\mathbb{F}(x_1, \dots, x_n) \rightarrow \text{SINGULAR}$

Motivation: Matching, Matroids (Lovász)

- Randomized poly-time
- [Main Open Q.] Deterministic Algorithm?
 $\rightarrow$ Ckt Lower Bound (KIO4)

NSINGULAR PROBLEM

- $X = \{x_1, \dots, x_n\} \rightarrow$ Noncommuting variables
- Rank in Free Skew Field $\rightarrow \mathbb{F}\langle x_1, \dots, x_n \rangle$

$$\begin{array}{c} \mathbb{F}\langle x_1, \dots, x_n \rangle \\ \uparrow \\ \mathbb{F}\langle x_1, \dots, x_n \rangle \end{array}$$

[Cohn, Amitsur]

C-Rank / NC-Rank

$$M = \begin{bmatrix} 0 & 1 & x_1 \\ -1 & 0 & x_2 \\ -x_1 & -x_2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & x_2x_1 - x_1x_2 \end{bmatrix}$$

$\Rightarrow$ C-rank = 2, NC-rank = 3,

Known Theorem

Thm [GAOW'16, IQS'18, HH21]:
 $\text{NSINGULAR} \in \text{P}.$

- $\rightarrow$ No direct connection with traditional PIT methods.

Key Concepts

Inner Rank:

$$\text{nr-rank}(T) = \min_{s, t} r \text{ s.t.}$$

$$T = P_{s \times r} Q_{r \times t}$$

Blow-up spaces

$$T = A_0 + \sum_{i=1}^n A_i x_i$$

$$\underline{p} = (p_1, p_2, \dots, p_n) \in M_d(\mathbb{F})^n$$

$$T(\underline{p}) = A_0 \otimes I_d + \sum_{i=1}^n A_i \otimes M_i$$

$$T^{\{d\}} = \left\{ T(\underline{p}) : \underline{p} \in M_d(\mathbb{F})^n \right\}$$

Key Fact [DM, IQS]

- [Regularity Lemma]: The maximum rank in $T^{\{d\}}$ is a multiple of d .

• Defⁿ: $\text{nc-rank}(T) = \lim_{d \rightarrow \infty} \frac{\text{rank}(T^{\{d\}})}{d}$
= Inner Rank.

Concept of Witness:

$\underline{p} = (p_1, \dots, p_n) \in M_d^n(\mathbb{F})$ is witness of $\text{nc-rank}(T) \geq r$ if $\text{rank}(T(\underline{p})) \geq rd$.

→ Key idea: Find better and better witness.

[Sketch] : NSINGULAR $\in P$.

Input : $T = A_0 + \sum A_i x_i$

Output : $\text{ncrank}(T)$.

→ Suppose already have witness $\mathcal{P}$ of rank = r
in d -dimension.

① Is r the maximum rank?

If "YES" → $\text{nc-Rank} = r$ [STOP].

② Otherwise : Use a "rank-increment"
step to produce witness of rank $> r$

③ [Rounding] : Another witness of rank $\geq r+1$

④ [Blow-up Control] : Keep the dimension small.

IDS : [Finite dimensional
division algebra, simple
linear algebra]

> A new insight.

nc-PIT - based "rank increment"

- Let $(p_1, \dots, p_n)$ be a rank witness p of dimension $= d$

Shift:

$z_1, \dots, z_n$: generic matrices

$$T_d(z_1 + p_1, \dots, z_n + p_n)$$

$$\rightarrow \cup \left[\begin{array}{c|c} I_{rd} - L & 0 \\ \hline 0 & C - B(I_{rd} - L)^{-1}A \end{array} \right] V$$

$\rightarrow$ Conceptually similar: [BBJP'19]

Key Observation

- rank can be increased

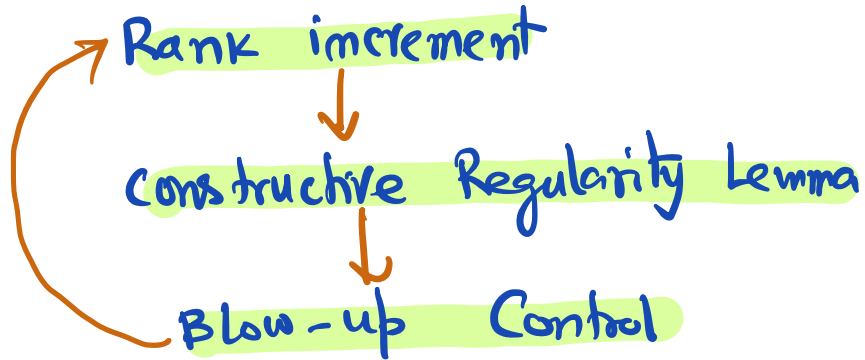
$$\Leftrightarrow C - B(I_{rd} - L)^{-1}A \neq 0$$

$$\Leftrightarrow C - B \left(\sum_{k \leq rd} L^k \right) A \neq 0$$

$\nearrow$
[Schützenberger]

- Final step: $\rightarrow$ nc ABP PIT
[Raz-Shpilka'05]

Steps:



Trade-off Result

$$X = X_1 \sqcup X_2 \sqcup \dots \sqcup X_k$$

X_i : Noncommuting

$$i \neq j : [X_i, X_j] = 0$$

Main Question:

- Given a linear matrix T over X ,
compute the rank of T .

$\uparrow U_{[k]} : [KV'20]$

$$\mathbb{F} \langle X_1 \sqcup X_2 \sqcup \dots \sqcup X_n \rangle$$

Main Theorem 1

A polynomial-time algorithm for the [PC]-SING problem for $k = O(1)$, $\mathbb{F} = \mathbb{Q}$.

Algebraic Question: $\leftarrow$ A central problem in algebraic automata theory

- Let $[T_1]_{s \times s}, [T_2]_{s \times s}$ are linear matrices defined over $X = X_1 \cup \dots \cup X_k$, $k = O(1)$, u_1, v_1, u_2, v_2 : vectors.

$$\text{check: } \vec{u}_1 \left(\sum_{i \geq 0} T_1^i \right) \vec{v}_1 \stackrel{?}{=} \vec{u}_2 \left(\sum_{i \geq 0} T_2^i \right) \vec{v}_2$$

Solution: In deterministic polynomial-time.

Example: $X_1 = \{x_1, x_2\}$, $X_2 = \{x'_1, x'_2\}$

$$\Rightarrow x_1 x'_2 x_2 x'_1 \sim x_1 x_2 x'_2 x'_1$$

History

- Rabin & Scott - 1959
- Griffiths - 1968
- Bird - 1973
- Valiant - 1974
- Friedman-Greibach - 1982
- Harju - Karhumäki - 1991
- Norrell - 2013 - Randomized Poly-time

A Finite Truncation [Norrell, Schützenberger]



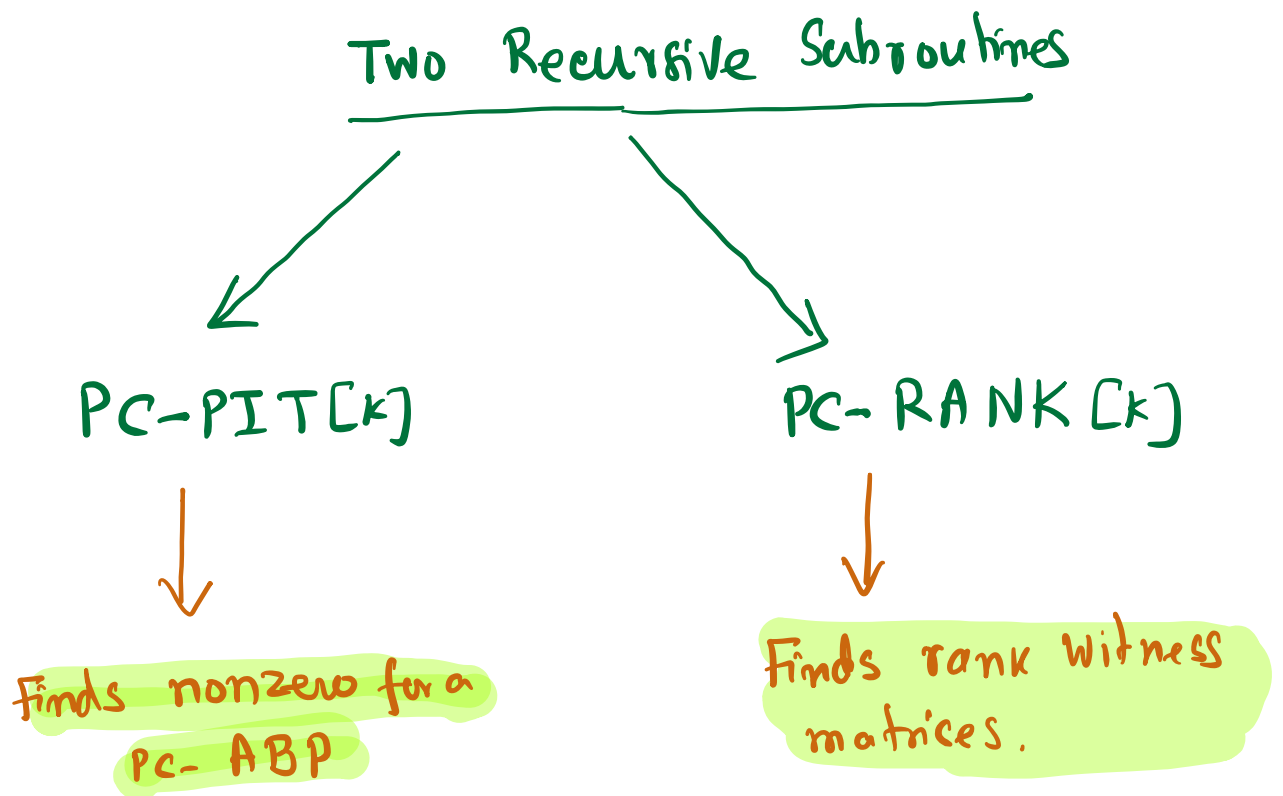
An ABP identity testing problem over X

(Glimpse of Proof Idea)

- Main point: The new NSINGULAR can be lifted in this setting.

↳ Do not know how to use other algorithms!

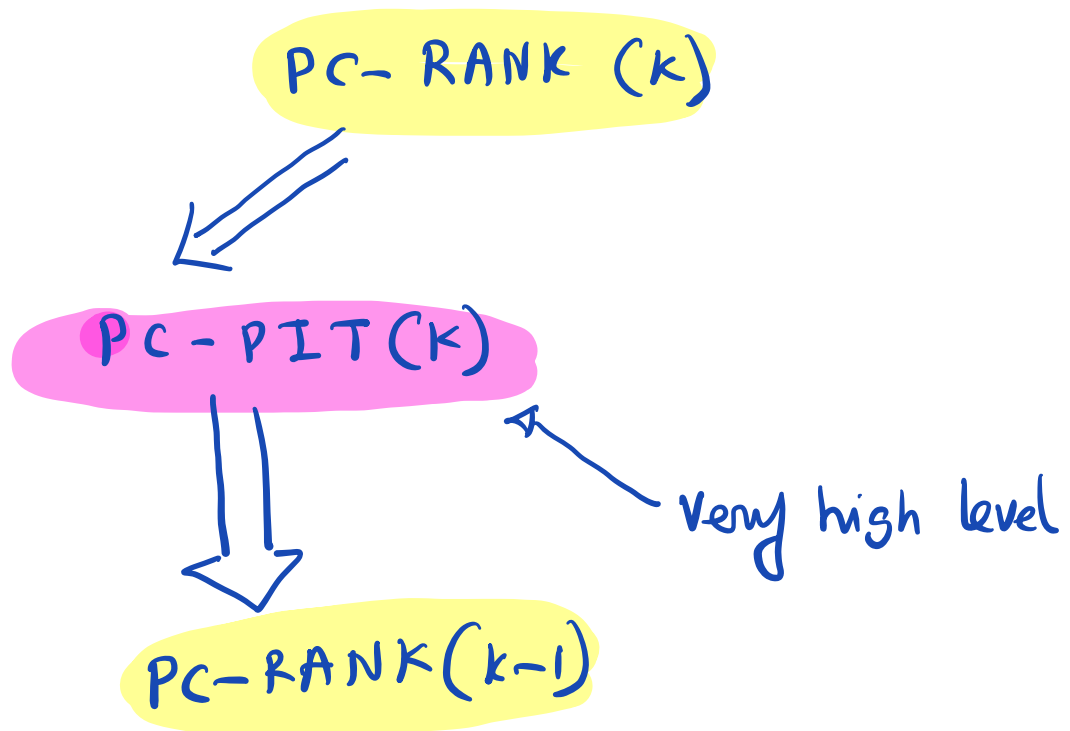
↳ particularly PIT connection



Sketch Contd.

$$\begin{array}{lcl} X_1 & \rightarrow & M_1 \otimes I_{d_2} \otimes I_{d_3} \otimes \dots \otimes I_{d_k} \\ X_2 & \rightarrow & I_{d_1} \otimes M_2 \otimes I_{d_3} \otimes \dots \otimes I_{d_k} \\ & \vdots & \\ X_k & \rightarrow & I_{d_1} \otimes I_{d_2} \otimes \dots \otimes M_k \end{array}$$

partially - commutative structure.



- Rounding and Blow-up Control

→ Somewhat sophisticated than

IQS'18

Some Remarks

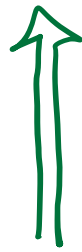
- Run time — $\delta^2 O(k \log k)$
↳ Wotfel : Randomized $\delta O(k)$

Improve Run time.

- For non constant k : No hardness result is known (for the series equivalence problem)
- No invariant theory connection known. [MW'20]

A central Open Question:

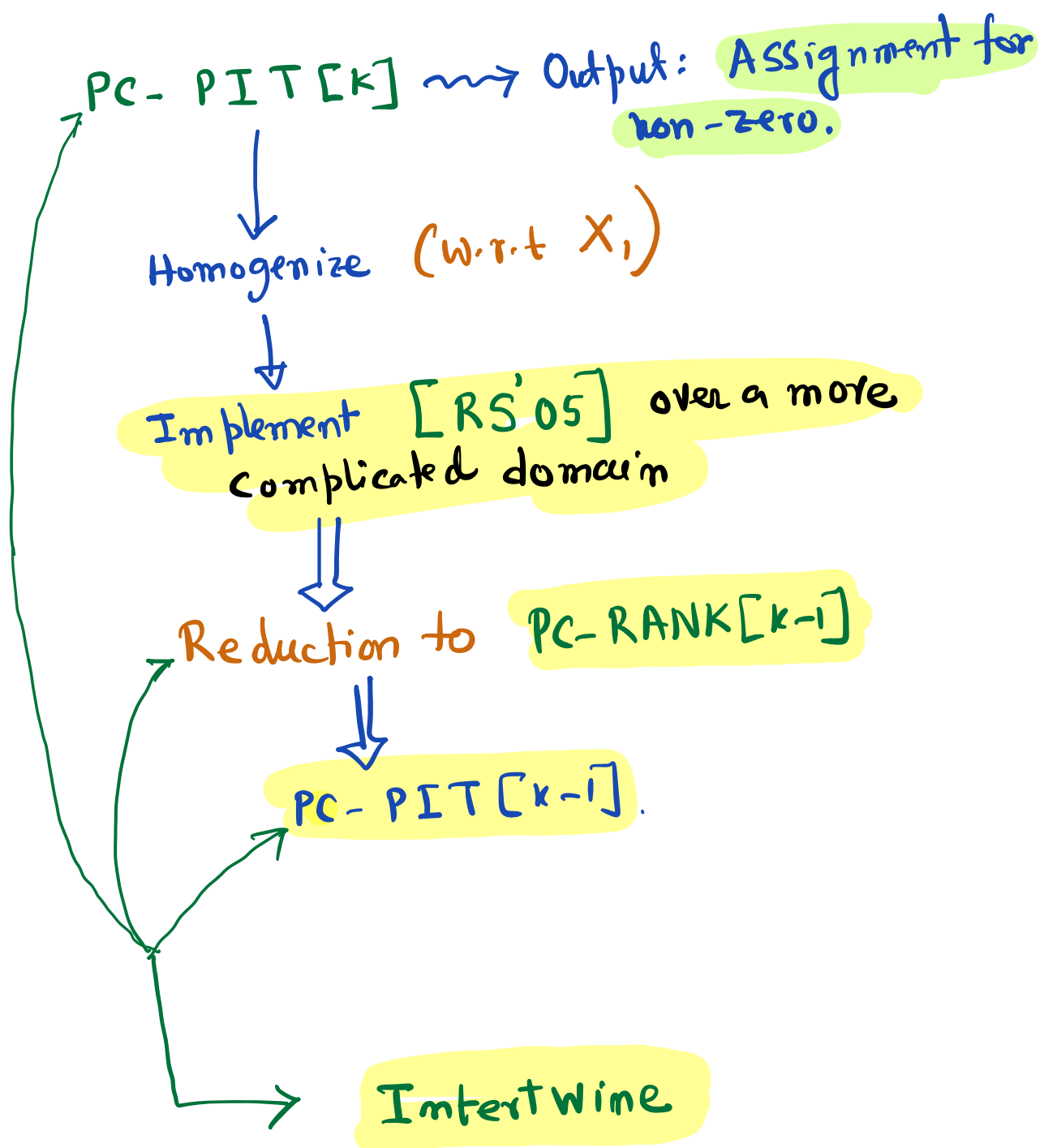
— Design efficient Black-box
algorithm for NSINGULAR



Quasi-NC algorithm
for Bipartite PM

[FQT'21]

More Sketch



PC-RANK [K]

↳ Output: $I_{d_1} \otimes I_{d_2} \otimes \dots \otimes M_x \otimes \dots \otimes I_{d_n}$



PC-PIT [K]

→ Used in Rank increment step.

Blow-Up Control and Rounding
Steps:

↳ Conceptually [IQS]

More work to implement.

Thank You!