# Sequences of three-tensors and algorithms for hard problems

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# Outline

- Based on joint work with **Andreas Björklund** (Lund), **Radu Curticapean** (Copenhagen & Regensburg), **Thore Husfeldt** (Copenhagen), **Tomohiro Koana** (Kyoto), **Mateusz Michałek** (Konstanz), **Jesper Nederlof** (Utrecht), **Kevin Pratt** (New York)

- Talk outline:
  1. Tensors, tensor rank, asymptotic rank / tensor exponents
  2. The asymptotic rank conjecture
     - The set cover conjecture is inconsistent with the asymptotic rank conjecture (STOC'24)
     - A faster deterministic algorithm for chromatic number under the asymptotic rank conjecture (SODA'25)
  3. A universal sequence of tensors for the asymptotic rank conjecture (ITCS'25)
  4. Work in progress:
     *Kronecker scaling* in sequences, the balanced tripartitioning tensors, the permanent

# Preliminaries: Tensors

- For a nonnegative integer $d$, we write $[d] = \{0, 1, \ldots, d-1\}$
- We work in coordinates, all tensors have order three
- An element $T \in \mathbb{C}^{d \times d \times d}$ is a **tensor** of **shape** $d \times d \times d$
- For $i, j, k \in [d]$, we write $T_{i,j,k}$ for the **entry** of $T$ at **position** $(i, j, k)$
- *Example.*
  The $4 \times 4 \times 4$ tensor $\mathrm{MM}_2$ is displayed below:

$$\mathrm{MM}_2 = \left[\begin{array}{cccc|cccc|cccc|cccc} 1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1 \end{array}\right]$$

# Preliminaries: Tensor rank

- A tensor $T \in \mathbb{C}^{d \times d \times d}$ has **rank one** if there exist three nonzero vectors $a, b, c \in \mathbb{C}^d$ such that $T_{i,j,k} = a_i b_j c_k$ for all $i, j, k \in [d]$
- The **rank** $\mathrm{R}(T)$ of a tensor $T \in \mathbb{C}^{d \times d \times d}$ is the least nonnegative integer $r$ such that $T$ can be written as a sum of $r$ rank one tensors
- *Example.* The rank of $\mathrm{MM}_2$ is $7$

# Preliminaries: Kronecker product and Kronecker powers

- Let $S \in \mathbb{C}^{d\times d\times d}$ and $T \in \mathbb{C}^{e\times e\times e}$ be tensors
- The **Kronecker product** $S \boxtimes T \in \mathbb{C}^{de\times de\times de}$ is defined for all $i, j, k \in [d]$ and $u, v, w \in [e]$ by

$$(S \boxtimes T)_{ie+u, je+v, ke+w} = S_{i,j,k} T_{u,v,w}$$

- For $S \in \mathbb{C}^{d\times d\times d}$ and a positive integer $p$, we write $S^{\boxtimes p} \in \mathbb{C}^{d^p\times d^p\times d^p}$ for the Kronecker product of $p$ copies of $S$
- We say that $S^{\boxtimes p}$ is the $p^{\text{th}}$ **Kronecker power** of $S$

# Example: Kronecker powers

$$P = \left[\begin{array}{c|c} 0\,1 & 1\,0 \\ 1\,0 & 0\,0 \end{array}\right]$$

$$P^{\boxtimes 2} = \left[\begin{array}{c|c|c|c} 0\,0\,0\,1 & 0\,0\,1\,0 & 0\,1\,0\,0 & 1\,0\,0\,0 \\ 0\,0\,1\,0 & 0\,0\,0\,0 & 1\,0\,0\,0 & 0\,0\,0\,0 \\ 0\,1\,0\,0 & 1\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \\ 1\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 \end{array}\right]$$

$$P^{\boxtimes 3} = \left[\begin{array}{c|c|c|c|c|c|c|c} 00000001 & 00000010 & 00000100 & 00001000 & 00010000 & 00100000 & 01000000 & 10000000 \\ 00000010 & 00000000 & 00001000 & 00000000 & 00100000 & 00000000 & 10000000 & 00000000 \\ 00000100 & 00001000 & 00000000 & 00000000 & 01000000 & 10000000 & 00000000 & 00000000 \\ 00001000 & 00000000 & 00000000 & 00000000 & 10000000 & 00000000 & 00000000 & 00000000 \\ 00010000 & 00100000 & 01000000 & 10000000 & 00000000 & 00000000 & 00000000 & 00000000 \\ 00100000 & 00000000 & 10000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 \\ 01000000 & 10000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 \\ 10000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 & 00000000 \end{array}\right]$$

# Tensor exponents and asymptotic rank

- The **exponent** $\sigma(T)$ of a tensor $T \in \mathbb{C}^{d\times d\times d}$ is the infimum of all $\sigma > 0$ such that $\mathrm{R}(T^{\boxtimes p}) \leq d^{\sigma p + o(p)}$ holds
- Equivalently, the **asymptotic rank** of $T$ is $\tilde{\mathrm{R}}(T) = \lim_{p\to\infty} \mathrm{R}(T^{\boxtimes p})^{1/p} = d^{\sigma(T)}$
- Exponents of *constant-size* tensors are fundamental to the study of algorithms
  - The exponent $\omega$ of square matrix multiplication satisfies $\omega = 2\sigma(\mathrm{MM}_2)$ [Strassen 1986/1988]
  - The set cover conjecture fails if the exponent of a specific $7 \times 7 \times 7$ tensor is sufficiently close to $1$ [Björklund & K. 2024] (see also [Pratt 2024])
  - If specific large but constant-size tensors have their exponents sufficiently close to $1$, then the chromatic number of a given $n$-vertex graph can be computed in $O(1.99982^n)$ time [Björklund, Curticapean, Husfeldt, K., & Pratt 2025]

# Example: The $7 \times 7 \times 7$ tensor [Björklund & K. 2024]

$$Q_7 = \left[\begin{array}{c|c|c|c|c|c|c}
0000000 & 0000001 & 0000010 & 0000100 & 0001000 & 0010000 & 0100000 \\
0000001 & 0000000 & 0000100 & 0000000 & 0010000 & 0000000 & 1000000 \\
0000010 & 0000100 & 0000000 & 0000000 & 0100000 & 1000000 & 0000000 \\
0000100 & 0000000 & 0000000 & 0000000 & 1000000 & 0000000 & 0000000 \\
0001000 & 0010000 & 0100000 & 1000000 & 0000000 & 0000000 & 0000000 \\
0010000 & 0000000 & 1000000 & 0000000 & 0000000 & 0000000 & 0000000 \\
0100000 & 1000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000
\end{array}\right]$$

If $\sigma(S) \leq 1.001$ then the Set Cover Conjecture is false

## Strassen's theory—the asymptotic spectrum

- Wigderson and Zuiddam give a comprehensive modern survey and development: https://www.math.ias.edu/~avi/PUBLICATIONS/WigdersonZu_Final_Draft_Oct2023.pdf

# The asymptotic rank conjecture

- Define the **worst-case** tensor exponent for $d \times d \times d$ tensors by

$$\sigma(d) = \sup_{T \in \mathbb{C}^{d \times d \times d}} \sigma(T)$$

- It is immediate that $\sigma(1) = 1$; it is a nontrivial consequence of the geometry of tensors that $\sigma(2) = 1$; already $\sigma(3)$ is unknown—it is known that $\sigma(3) = 1$ implies $\omega = 2$
- Strassen (1988, implicit) has shown that $\sigma(d) \leq 2\omega/3$ for all $d \geq 1$; the following bold conjecture has been made by many
- **Conjecture. (Asymptotic rank conjecture)**
  For all $d \geq 1$ it holds that $\sigma(d) = 1$
- [Strassen (1994) has conjectured $\sigma(T) = 1$ for tight and concise tensors $T$.]

# But how to approach the asymptotic rank conjecture?

# Universal objects for the asymptotic rank conjecture?

- Strassen's result $\sigma(d) \leq 2\omega/3$ can be rewritten as $\sigma(d) \leq \frac{4}{3}\sigma(\mathrm{MM}_2)$
- Could one identify an explicit "universal" tensor $U_d$ with $\sigma(d) = \sigma(U_d)$ ?

# A universal sequence of tensors [K. & Michałek 2025]

- We present an explicit **sequence** $\mathcal{U}_d$ of tensors that is universal and consists of zero-one-valued tensors
- Towards this end, let us extend the definition of the exponent of a tensor $T$ to a **sequence** $\mathcal{T} = (T_j \in \mathbb{C}^{s_j \times s_j \times s_j} : j = 1, 2, \ldots)$ of nonzero tensors
- The **exponent** $\sigma(\mathcal{T})$ of $\mathcal{T}$ is the infimum of all $\sigma > 0$ such that $\mathrm{R}(T_j) \leq s_j^{\sigma + o(1)}$ holds
- *Remark.*
  The exponent $\sigma(S)$ of $S \in \mathbb{C}^{d \times d \times d}$ agrees with the exponent $\sigma(\mathcal{S})$ of the **Kronecker power sequence** $\mathcal{S} = (S^{\boxtimes q} : q = 1, 2, \ldots)$
- **Theorem (A universal sequence of tensors for fixed $d$).**
  For all $d = 1, 2, \ldots$ there is an explicit sequence $\mathcal{U}_d$ with $\sigma(\mathcal{U}_d) = \sigma(d)$

# Proof ideas/sketch — the Kronecker power map $\mathrm{K}_{d,q}$

- For positive integers $d$ and $q$, the **Kronecker power map**

$$\mathrm{K}_{d,q} : \mathbb{C}^{d\times d\times d} \to \mathbb{C}^{d^q\times d^q\times d^q}$$

  takes a tensor $S \in \mathbb{C}^{d\times d\times d}$ to its $q^{\text{th}}$ Kronecker power $\mathrm{K}_{d,q}(S) = S^{\boxtimes q}$
- For a tensor $T \in \mathbb{C}^{d^q\times d^q\times d^q}$, it is convenient to index the entries $T_{I,J,K} \in \mathbb{C}$ with $q$-tuples $I = (i_1, i_2, \ldots, i_q) \in [d]^q$, $J = (j_1, j_2, \ldots, j_q) \in [d]^q$, $K = (k_1, k_2, \ldots, k_q) \in [d]^q$
- The image $\mathrm{K}_{d,q}(S) = S^{\boxtimes q}$ of a tensor $S \in \mathbb{C}^{d\times d\times d}$ is defined entrywise for all $I, J, K \in [d]^q$ by

$$S^{\boxtimes q}_{I,J,K} = S_{i_1,j_1,k_1} S_{i_2,j_2,k_2} \cdots S_{i_q,j_q,k_q}$$

- The linear span of the image $K_{d,q}(\mathbb{C}^{d^q\times d^q\times d^q})$ has dimension $I_{d,q} = \binom{d^3-1+q}{d^3-1}$, which grows *only polynomially in* $q$ when $d$ is constant
- The linear span has an explicit zero-one-valued basis $B^{(d,q,i)} \in \mathbb{C}^{d^q\times d^q\times d^q}$ with $i \in [I_{d,q}]$
- Take $U^{(d,q)} = \oplus_{i\in[I_{d,q}]} B^{(d,q,i)}$ to get the universal sequence $\mathcal{U}_d = \{U^{(d,q)} : q = 1, 2, \ldots\}$

Work point:

The tensors $U^{(d,q)}$ are invariant under a diagonal action of $S_q$

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Can use representation theory to approach the asymptotic rank conjecture ?

Work point:

Are there other natural sequences of tensors than the Kronecker power sequence $S, S^{\boxtimes 2}, S^{\boxtimes 3}, \ldots$ of a tensor $S$ ?

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Sequences that are "almost" Kronecker power sequences?

# Work in progress (1/2) — [Björklund, K., Koana, Nederlof]

- A tensor sequence $\mathcal{T} = (T_n \in \mathbb{C}^{s_n \times s_n \times s_n} : n = 1, 2, \ldots)$ has the **Kronecker scaling property** if for all $\delta > 0$ there exist infinitely many $d = 1, 2, \ldots$ such that for all large enough $n = 1, 2, \ldots$ in an arithmetic progression the tensor $T_n$ is a sum of at most $2^{\delta n}$ tensors, each of which is a restriction of $T_d^{\boxtimes s}$ for $s \leq (1 + \delta)n/d$
- *Example.* Kronecker power sequences have the Kronecker scaling property
- **Theorem.** [Björklund, K., Koana, Nederlof]
  The tensor sequence $\mathcal{P} = (P_n \in \mathbb{C}^{\binom{3n}{n} \times \binom{3n}{n} \times \binom{3n}{n}} : n = 1, 2, \ldots)$ consisting of the **balanced tripartitioning** tensors

$$P_n(x, y, z) = \sum_{\substack{A, B, C \in \binom{[3n]}{n} \\ A \cup B \cup C = [3n]}} x_A y_B z_C$$

  has the Kronecker scaling property
- *Remark.* It is known that $1 \leq \sigma(\mathcal{P}) \leq H(1/3)^{-1}$, where $H$ is the binary entropy function

# Work in progress (2/2) — [Björklund, K., Koana, Nederlof]

- The **permanent** of a square matrix $A \in \mathbb{C}^{n\times n}$ is $\operatorname{per} A = \sum_{\pi \in S_n} \prod_{i\in[n]} A_{i,\pi(i)}$
- The best general algorithm known is due to Ryser (1963), who presented a simple inclusion–exclusion formula that can be used to compute the permanent with $O(2^n n)$ arithmetic operations
- **Theorem.** [Björklund, K., Koana, Nederlof]
  For all $\epsilon > 0$ there exists an algorithm that given $n$ as input runs in time $O\left(2^{H(1/3)(\sigma(\mathcal{P})+\epsilon)n}\right)$ and outputs an arithmetic circuit of size $O\left(2^{H(1/3)(\sigma(\mathcal{P})+\epsilon)n}\right)$ for the $n \times n$ permanent
- *Remark.* It is open whether $\sigma(\mathcal{P}) < H(1/3)^{-1}$, but under the asymptotic rank conjecture it follows from Kronecker scaling that $\sigma(\mathcal{P}) = 1$

Work point:

Prove or disprove $\sigma(\mathcal{P}) < H(1/3)^{-1}$

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Can representation theory be used here?

# Summary

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