Low-depth algebraic circuit lower bounds over any field

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Appeared in CCC 2024 (*Best Paper*)

April 4, 2025

Theorem

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Let $\mathbb{F}$ be a field. There is an *n-variate degree-d polynomial*

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Let $\mathbb{F}$ be a field. There is an explicit n-variate degree-d polynomial

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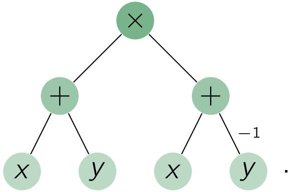
- *matches best known quantitative parameters [BDS22].*

Multivariate polynomials can be computed by small algebraic circuits, e.g.

$$x^2 - y^2 = (x + y)(x - y) \quad \sim$$



- size 7
- depth 2
- any $\mathbb{F}$
- (total) degree 2

The **size** is the number of nodes.

Goal (Algebraic Complexity Theory)

Find explicit polynomials requiring algebraic circuits of super-polynomial size.

parameters:

- depth: maximum length of input-output path
- $\mathbb{F}$: domain of constants appearing in circuit

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- applications of polynomial identities to algebraic complexity theory

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A polynomial is **homogeneous** if all monomials that appear have the same degree d. A circuit is **homogeneous** if all gates compute homogeneous polynomials.

e.g. the **elementary symmetric polynomial**

$$\mathrm{esym}_{n,d} = \sum_{S \subseteq [n], |S| = d} \prod_{i \in S} x_i \ ,$$

is homogeneous of degree d.

Definition

Let the variables be partitioned into $x_{1,1}, \ldots, x_{1,n}, \ \ldots \ , x_{d,1}, \ldots, x_{d,n} = \overline{x}_1, \ldots, \overline{x}_d$. A monomial is **set-multilinear** if is a product of one variable per $\overline{x}_i$, e.g. $\prod_{i=1}^{d} x_{i,j_i}$. A polynomial is **set-multilinear** if all monomials are set-multilinear. A circuit is **set-multilinear** if all gates compute set-multilinear polynomials.

e.g. the **rectangular permanent**

$$\mathrm{perm}_{n,d} = \sum_{\sigma : [d] \hookrightarrow [n]} \prod_{i=1}^{d} x_{i,\sigma(i)}$$

is set-multilinear. When $d = n$ this is the standard permanent.

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General circuits versus homogeneous circuits?

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Proof.

- use Newton identities relating $\mathrm{esym}_{n,d}$ and $\mathrm{pow}_{n,d} = \sum_{i=1}^{n} x_i^d$ ($\mathrm{char}(F) > d$)
- count integer partitions

$\square$

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- apply depth-4 homog circuit for $\mathrm{esym}_{D,d}$ to homogeneous $y_j \leftarrow \sum_{k=1}^{n} \beta_{j,k} x_k$ □

Question (LST22b, GHT22, FLST23)

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Compute $\mathrm{esym}_{n,d}$

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1. *I don't know.*

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Question

What **else** *can we do?*

Theorem

Let $\mathbb{F}$ be a field. There is an explicit n-variate degree-$\Theta(\log n)$ polynomial requiring size $n^{\Omega(\sqrt{\log n})}$ to be computed by depth-3 algebraic circuits over $\mathbb{F}$.

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Lemma

$p(\overline{x}) \in \mathbb{Z}[\overline{x}]$,

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$p(\overline{x}) \in \mathbb{Z}[\overline{x}]$, $\mathbb{F}$ any field.

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$p(\overline{x}) \in \mathbb{Z}[\overline{x}]$, $\mathbb{F}$ *any field.* $p(\overline{x}) = 0$

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$p(\overline{x}) \in \mathbb{Z}[\overline{x}]$, $\mathbb{F}$ *any field.* $p(\overline{x}) = 0$ *(in* $\mathbb{Z}[\overline{x}]$*)*

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$p(\overline{x}) \in \mathbb{Z}[\overline{x}]$, $\mathbb{F}$ any field. $p(\overline{x}) = 0$ (in $\mathbb{Z}[\overline{x}]$) $\implies$ $p(\overline{x}) = 0$,

Theorem

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$p(\overline{x}) \in \mathbb{Z}[\overline{x}]$, $\mathbb{F}$ *any field.* $p(\overline{x}) = 0$ *(in $\mathbb{Z}[\overline{x}]$)* $\Longrightarrow$ $p(\overline{x}) = 0$, *in* $\mathbb{F}[\overline{x}]$.

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$p(\overline{x}) \in \mathbb{Z}[\overline{x}]$, $\mathbb{F}$ any field. $p(\overline{x}) = 0$ (in $\mathbb{Z}[\overline{x}]$) $\implies$ $p(\overline{x}) = 0$, in $\mathbb{F}[\overline{x}]$.

- $\det(X)\det(Y) = \det(XY)$

Theorem

Let $\mathbb{F}$ be a field. There is an explicit n-variate degree-$\Theta(\log n)$ polynomial requiring size $n^{\Omega(\sqrt{\log n})}$ to be computed by depth-3 algebraic circuits over $\mathbb{F}$.

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Express LST as polynomial identity over $\mathbb{Z}$, then transfer identity to every $\mathbb{F}$.

Metafact

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Theorem

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$\Longrightarrow$ this f cannot have a small depth-3 ckt over any $\mathbb{F}$ $\square$

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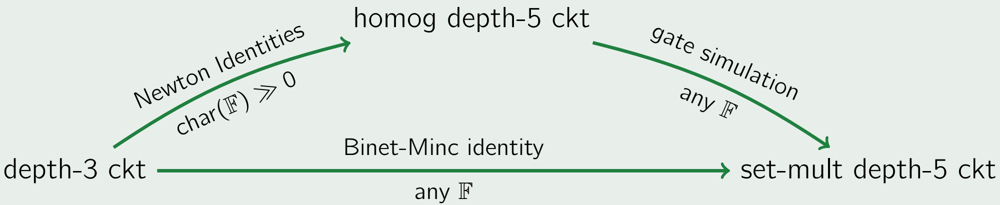

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Definition

Let the variables be partitioned into $\overline{x} = \overline{x}_1, \cdots, \overline{x}_d$. A monomial is **set-multilinear** if is a product of one variable per $\overline{x}_i$, e.g. $\prod_{i=1}^{d} x_{i,j_i}$. A polynomial is **set-multilinear** if all monomials are set-multilinear. A circuit is **set-multilinear** if all gates compute set-multilinear polynomials.

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Example

$\mathrm{perm}_{n,2}(\overline{x}, \overline{y})$

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  $\prod_{k=1}^{D}(\gamma_k + \sum_i \alpha_{k,i} x_i + \sum_j \beta_{k,j} y_j)$

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Corollary

size s depth-3 circuit

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size s depth-3 circuit $\xRightarrow{\text{any } \mathbb{F}}$ *depth-5 set-multilinear circuit of size* $\mathrm{poly}(s, d^d)$.

This talk:

- LST22 gave super-polynomial lower bounds against constant-depth algebraic circuits, in *large* characteristic fields
  - low-depth homogenization via the Newton identities, in *large* characteristic fields
  - low-depth set-multilinearization (of homogeneous circuits), over *any* field
  - strong lower bounds against constant-depth set-multilinear circuits, over *any* field
- **this work:** LST22 over *any* field
  - proof 1 (logical): proof LST22 is sufficiently "algebraic" so a proof in characteristic zero implies a proof over any field
  - proof 2 (constructive): low-depth set-multilinearization (of general circuits), via the Binet-Minc identity, over *any* field

Open Questions:

- low-depth homogenization over any field?

Thanks!

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