A brief survey on de-bordering paradigms and its recent advances

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WACT @ Ruhr-Universität Bochum

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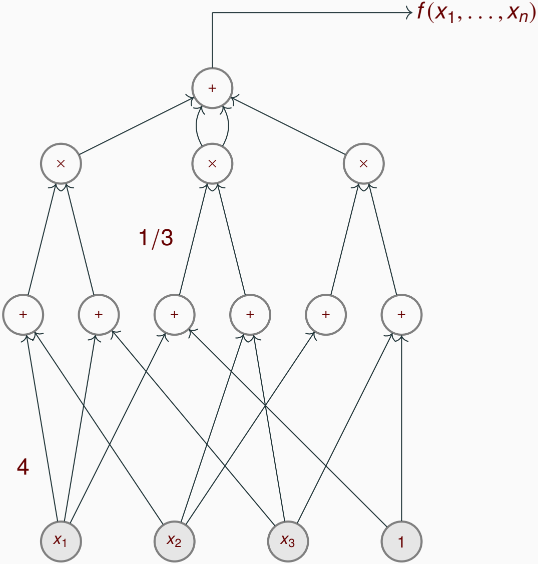
- VBP: The class $\mathsf{VBP} := \{(f_n(x_1, \ldots, x_m))_n \mid \ m, \mathrm{dc}(f_n) = \mathrm{poly}(n)\}$.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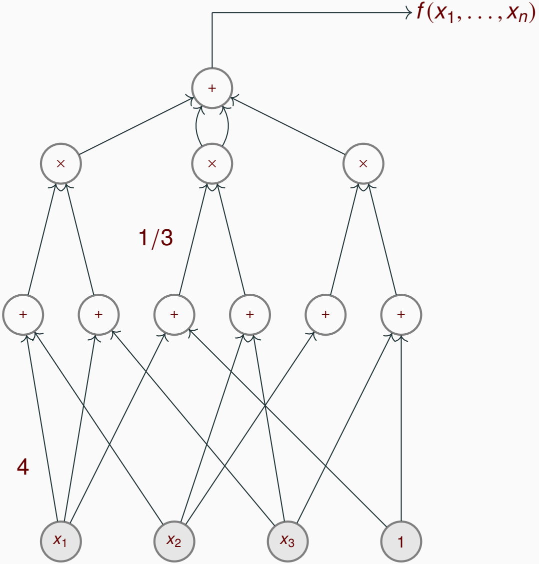
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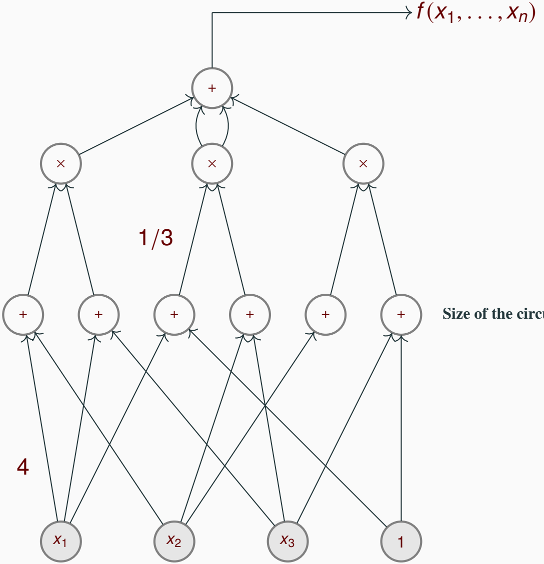

Size of the circuit = number of nodes + edges

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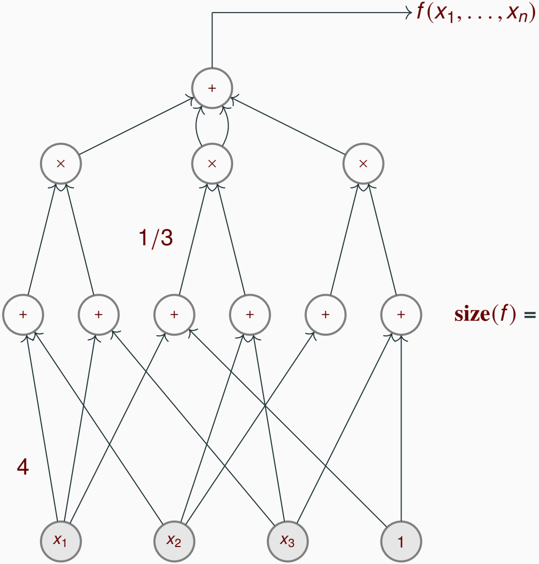

size(f) **= min size of the circuit computing** f

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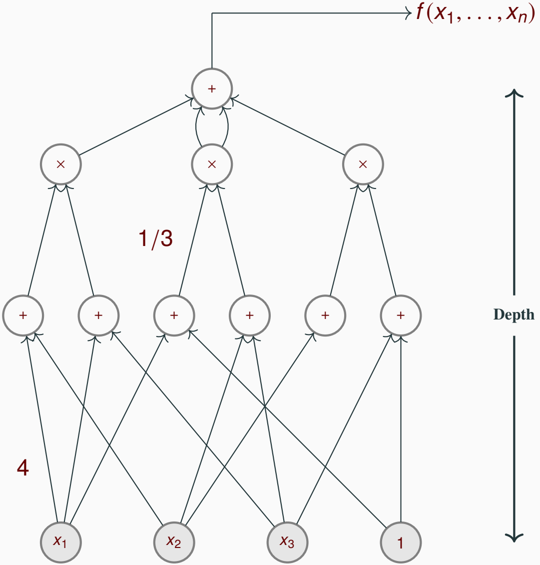

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$\mathsf{VNP} \not\subseteq \mathsf{VBP}, \mathsf{VNP} \not\subseteq \mathsf{VP}$. Equivalently, $\mathsf{dc}(\mathsf{perm}_n), \mathrm{size}(f_n) = n^{\omega(1)}$.

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  - $\mathsf{P/poly} \neq \mathsf{NP/poly} \implies \mathsf{VBP} \neq \mathsf{VNP}$ (over finite fields).
  - Assuming GRH (Generalized Riemann hypothesis), the results hold over $\mathbb{C}$ as well.
- $\mathsf{P/poly} = \mathsf{NP/poly} \implies \mathsf{PH} = \Sigma_2$ (i.e. Polynomial Hierarchy collapses) [Karp-Lipton 1980].

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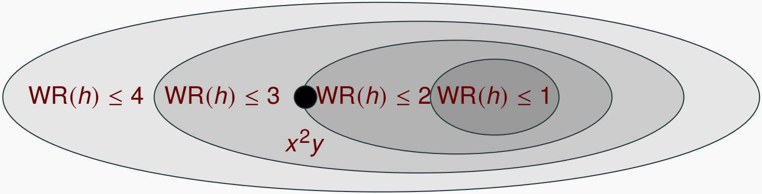

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- We *do not understand* the gap between the Waring rank and border Waring rank.

Border Waring Rank and Matrix Multiplication

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(ii) Show that $\overline{\mathsf{WR}}(x_1 \cdots x_n) = 2^{n-1}$ [We know $2^n/\sqrt{n}$, via partial-derivatives].

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- **Diagonalization trick** [Shpilka'25]: After suitable linear transformation and perturbation, x_i does not appear in $g_1, ..., g_{i-1}$.

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- **Open Question**. Can this be improved?

New Lower Bound Questions

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Strengthening Valiant's Conjecture [Milind-Sohoni 2001]

$\mathsf{VNP} \not\subset \overline{\mathsf{VBP}}, \mathsf{VNP} \not\subset \overline{\mathsf{VP}}$. Equivalently, $\overline{\mathsf{dc}}(\mathsf{perm}_n), \overline{\text{size}}(\mathsf{perm}_n) = n^{\omega(1)}$.

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- Every homogeneous degree d polynomial f can be written as a product

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$$\mathsf{VBP}_{[1]} \;:=\; \{(f_n)_n \mid f_n = \det(A_0 + \sum_{i=1}^{n} A_i x_i), A_i \in \mathbb{F}^{\mathrm{poly}(n)\times \mathrm{poly}(n)}, \mathrm{rank}(A_i) = 1\}\;.$$

- $\overline{\mathsf{VBP}_{[1]}} = \mathsf{VBP}_{[1]}$.

Open Question: $\overline{\mathsf{VBP}_{[2]}} \stackrel{?}{=} \mathsf{VBP}_{[2]}$.

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Chow rank and border Chow rank

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Chow Rank

Let $f \in S^d\mathbb{C}^n$. Chow rank of f, $\mathrm{CR}(f)$, is the smallest k such that f can be written as a sum of d-product of linear forms ℓ_i, i.e. $f = \sum_{i=1}^{k} \prod_{j=1}^{d} \ell_{i,j}$.

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- Does this hold for border?

De-bordering border Chow rank

Upper bound for $\overline{\mathsf{CR}}$ [Dutta-Dwivedi-Saxena'21].

Let $f \in S^d\mathbb{C}^n$, s.t. $\overline{\mathsf{CR}}(f) = k$. Then,

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100-ft above Idea for $\overline{\Sigma^{[2]}\Pi\Sigma}$: If $\lim_{\epsilon \to 0}(T_1 + T_2) = f$, where T_i are products of linear forms, then $\left(\frac{T_1}{T_2}\right)'$ has "nice" deboder-friendly expression.

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Hierarchy Theorem [Dutta-Saxena 2022]

Fix any constant $k \geq 1$. There is an explicit n-variate and $< n$ degree polynomial f such that f can be computed by an $O(n)$-size $\Sigma^{[k+1]}\Pi\Sigma$ circuit such that if f is computed by a $\overline{\Sigma^{[k]}\Pi\Sigma}$ circuit, then it requires size $2^{\Omega(n)}$.

Conclusion

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Thank you! Questions?