Uniformity for limits of tensors

arXiv 2305.19866
with Bik-Eggermont-Snowden

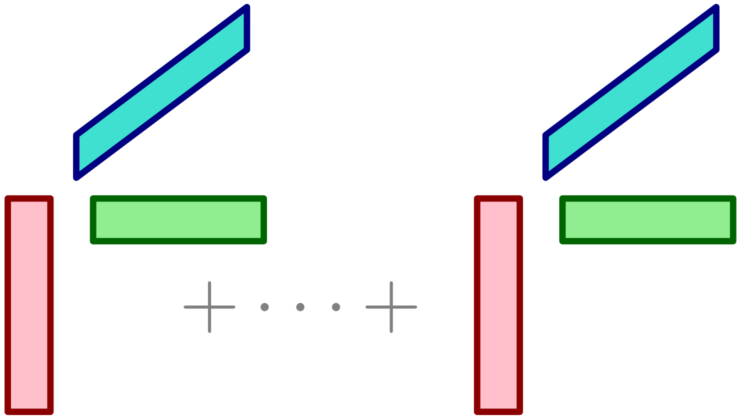
Jan Draisma
University of Bern

WACT
Bochum, April 2025

Some notions of rank

Tensor rank for $f \in V^{\otimes d}$:

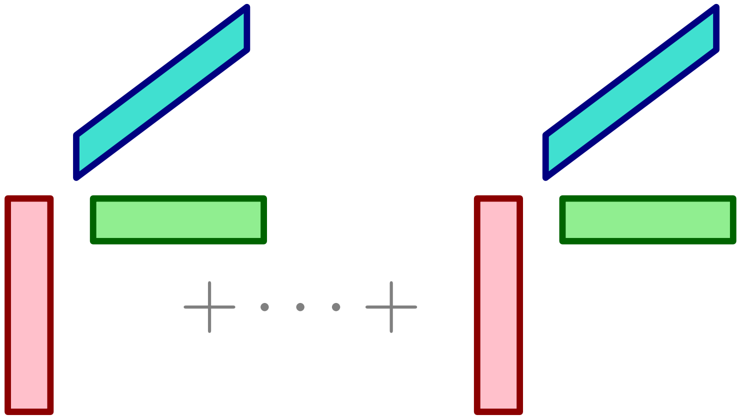
$\min\{r \mid f = \sum_{i=1}^{r} v_{i1} \otimes \cdots \otimes v_{id}\}$



Some notions of rank

Tensor rank for $f \in V^{\otimes d}$:

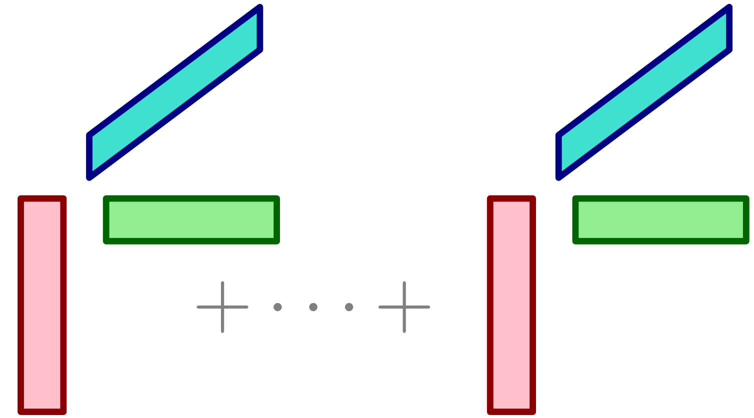
$\min\{r \mid f = \sum_{i=1}^{r} v_{i1} \otimes \cdots \otimes v_{id}\}$



Border rank:

$\min\{r \mid f = \lim_{\epsilon \to 0} \sum_{i=1}^{r} v_{i1}(\epsilon) \otimes \cdots \otimes v_{id}(\epsilon)\}$

Some notions of rank



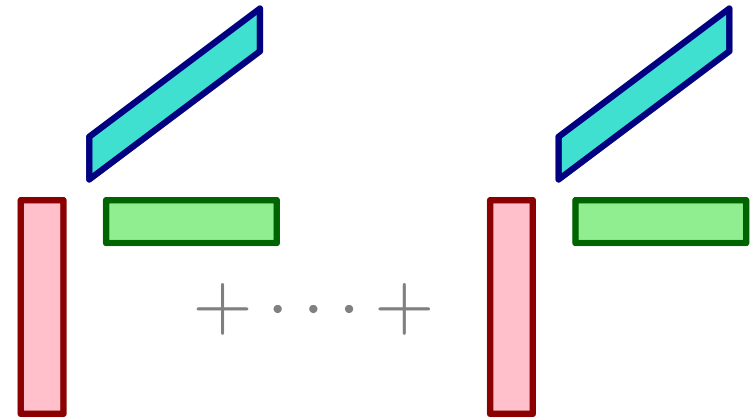
Tensor rank for $f \in V^{\otimes d}$:
$\min\{r \mid f = \sum_{i=1}^{r} v_{i1} \otimes \cdots \otimes v_{id}\}$

Border rank:
$\min\{r \mid f = \lim_{\epsilon \to 0} \sum_{i=1}^{r} v_{i1}(\epsilon) \otimes \cdots \otimes v_{id}(\epsilon)\}$

Quadric rank for $f \in S^4 V = \mathbb{C}[x_1, \ldots, x_n]_4$:
$\min\{r \mid f = \sum_{i=1}^{r} g_i h_i \text{ with } \deg(g_i) = \deg(h_i) = 2\}$

Some notions of rank

Tensor rank for $f \in V^{\otimes d}$:
$\min\{r \mid f = \sum_{i=1}^{r} v_{i1} \otimes \cdots \otimes v_{id}\}$



Border rank:
$\min\{r \mid f = \lim_{\epsilon \to 0} \sum_{i=1}^{r} v_{i1}(\epsilon) \otimes \cdots \otimes v_{id}(\epsilon)\}$

Quadric rank for $f \in S^4 V = \mathbb{C}[x_1, \dots, x_n]_4$:
$\min\{r \mid f = \sum_{i=1}^{r} g_i h_i \text{ with } \deg(g_i) = \deg(h_i) = 2\}$

Border quadric rank:
$\min\{r \mid f = \lim_{\epsilon \to 0} \sum_{i=1}^{r} g_i(\epsilon) h_i(\epsilon), \deg(g_i) = \deg(h_i) = 2\}$

Tensor varieties and tensor maps

Definition

closed

A *tensor variety* is a rule $X : V \mapsto X(V) \subseteq V^{\otimes d} =: T(V)$ s.t. for all linear $\varphi : V \to W$ we have $\varphi^{\otimes d} X(V) \subseteq X(W)$.

Definition

closed

A *tensor variety* is a rule $X : V \mapsto X(V) \subseteq V^{\otimes d} =: T(V)$ s.t. for all linear $\varphi : V \to W$ we have $\varphi^{\otimes d} X(V) \subseteq X(W)$.

Example: $X(V) = \{v_1 \otimes \cdots \otimes v_d \mid v_i \in V\} \subseteq T(V)$.

Definition

closed

A *tensor variety* is a rule $X : V \mapsto X(V) \subseteq V^{\otimes d} =: T(V)$ s.t. for all linear $\varphi : V \to W$ we have $\varphi^{\otimes d} X(V) \subseteq X(W)$.

Example: $X(V) = \{v_1 \otimes \cdots \otimes v_d \mid v_i \in V\} \subseteq T(V)$.

More generally: $T \rightsquigarrow$ a sum of Schur functors, $\varphi^{\otimes d} \rightsquigarrow T(\varphi)$.
Write $X(\varphi) := T(\varphi)|_{X(V)} : X(V) \to X(W)$.

Tensor varieties and tensor maps

Definition

closed

A *tensor variety* is a rule $X : V \mapsto X(V) \subseteq V^{\otimes d} =: T(V)$ s.t. for all linear $\varphi : V \to W$ we have $\varphi^{\otimes d} X(V) \subseteq X(W)$.

Example: $X(V) = \{v_1 \otimes \cdots \otimes v_d \mid v_i \in V\} \subseteq T(V)$.

More generally: $T \rightsquigarrow$ a sum of Schur functors, $\varphi^{\otimes d} \rightsquigarrow T(\varphi)$.
Write $X(\varphi) := T(\varphi)|_{X(V)} : X(V) \to X(W)$.

Definition

A *morphism* $\alpha : X \to Y$ of tensor varieties is a rule $V \mapsto \alpha_V : X(V) \to Y(V)$ s.t. $\forall \varphi : V \to W$:

morphism

$$\begin{array}{ccc} X(V) & \xrightarrow{\alpha_V} & Y(V) \\ {\scriptstyle X(\varphi)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle Y(\varphi)} \\ X(W) & \xrightarrow{\alpha_W} & Y(W) \end{array}$$

Image closure of morphisms

Easy fact: $\alpha : X \to Y$ a morphism $\rightsquigarrow \overline{\mathrm{im}(\alpha)} : V \mapsto \overline{\mathrm{im}(\alpha_V)}$ is a tensor subvariety of Y.

Central challenge: describe elements in $\overline{\mathrm{im}(\alpha)}$ uniformly.

Image closure of morphisms

Easy fact: $\alpha : X \to Y$ a morphism $\rightsquigarrow \overline{\mathrm{im}(\alpha)} : V \mapsto \overline{\mathrm{im}(\alpha_V)}$ is a tensor subvariety of Y.

Central challenge: describe elements in $\overline{\mathrm{im}(\alpha)}$ uniformly.

Example [Ballico-Bik-Oneto-Ventura, 2022]

$X(V) = (S^2V)^6$ (six quadrics), $Y(V) = S^4V$ (one quartic)

$\alpha_V(g_1, h_1, g_2, h_2, g_3, h_3) := g_1h_1 + g_2h_2 + g_3h_3$.

Note: $\mathrm{im}(\alpha_V) = \{$quartics of quadric rank $\leq 3\}$.

Image closure of morphisms

Easy fact: $\alpha : X \to Y$ a morphism $\rightsquigarrow \overline{\text{im}(\alpha)} : V \mapsto \overline{\text{im}(\alpha_V)}$ is a tensor subvariety of Y.

Central challenge: describe elements in $\overline{\text{im}(\alpha)}$ uniformly.

Example [Ballico-Bik-Oneto-Ventura, 2022]

$X(V) = (S^2 V)^6$ (six quadrics), $Y(V) = S^4 V$ (one quartic)

$\alpha_V(g_1, h_1, g_2, h_2, g_3, h_3) := g_1 h_1 + g_2 h_2 + g_3 h_3$.

Note: $\text{im}(\alpha_V) = \{$quartics of quadric rank $\leq 3\}$.

But $\overline{\text{im}(\alpha_V)}$ also contains $\lim_{\epsilon \to 0}$
$\frac{1}{\epsilon}[(x^2 + \epsilon g)(y^2 + \epsilon f) - (u^2 - \epsilon q)(v^2 - \epsilon p) - (xy + uv)(xy - uv)] =$
$x^2 f + y^2 g + u^2 p + v^2 q =: \beta_V(x, y, u, v, f, g, p, q)$.

Image closure of morphisms

Easy fact: $\alpha : X \to Y$ a morphism $\rightsquigarrow \overline{\mathrm{im}(\alpha)} : V \mapsto \overline{\mathrm{im}(\alpha_V)}$ is a tensor subvariety of Y.

Central challenge: describe elements in $\overline{\mathrm{im}(\alpha)}$ uniformly.

Example [Ballico-Bik-Oneto-Ventura, 2022]

$X(V) = (S^2 V)^6$ (six quadrics), $Y(V) = S^4 V$ (one quartic)

$\alpha_V(g_1, h_1, g_2, h_2, g_3, h_3) := g_1 h_1 + g_2 h_2 + g_3 h_3$.

Note: $\mathrm{im}(\alpha_V) = \{$quartics of quadric rank $\leq 3\}$.

But $\overline{\mathrm{im}(\alpha_V)}$ also contains $\lim_{\epsilon \to 0}$ $\frac{1}{\epsilon}[(x^2 + \epsilon g)(y^2 + \epsilon f) - (u^2 - \epsilon q)(v^2 - \epsilon p) - (xy + uv)(xy - uv)] = x^2 f + y^2 g + u^2 p + v^2 q =: \beta_V(x, y, u, v, f, g, p, q)$.

Theorem [BBOV]: $\mathrm{im}(\alpha_V)$ is not closed for $\dim(V) \gg 0$.

Main result

Older theorem [Bik-D-Eggermont-Snowden, 2021]
Y any tensor variety, then $\exists$ irreducible affine varieties B_i, sums T_i of Schur functors, and morphisms
$\beta_i : B_i \times T_i \to Y, i = 1, \ldots, k$, such that $Y = \bigcup_{i=1}^{k} \mathrm{im}(\beta_i)$.

Main result

Older theorem [Bik-D-Eggermont-Snowden, 2021]

Y any tensor variety, then $\exists$ irreducible affine varieties B_i, sums T_i of Schur functors, and morphisms $\beta_i : B_i \times T_i \to Y, i = 1, \ldots, k$, such that $Y = \bigcup_{i=1}^{k} \mathrm{im}(\beta_i)$.

Uses *Noetherianity* of tensor varieties and *unirationality*: if Y is irreducible, then $\exists$ dominant $\beta : B \times T \to Y$.

Main result

Older theorem [Bik-D-Eggermont-Snowden, 2021]
Y any tensor variety, then $\exists$ irreducible affine varieties B_i, sums T_i of Schur functors, and morphisms $\beta_i : B_i \times T_i \to Y, i = 1, \ldots, k$, such that $Y = \bigcup_{i=1}^{k} \mathrm{im}(\beta_i)$.

Uses *Noetherianity* of tensor varieties and *unirationality*: if Y is irreducible, then $\exists$ dominant $\beta : B \times T \to Y$.

Main Theorem [BDES, 2023]
$\alpha : X \to Y$ a morphism of tensor varieties, then there is an $N \in \mathbb{N}$ such that for all V and all $y \in \overline{\mathrm{im}(\alpha_V)}$ there is a formal curve $x(\epsilon) \in X(V)(\mathbb{C}((\epsilon)))$ with exponents $\geq -N$ such that $y = \lim_{\epsilon \to 0} \alpha_V(x(\epsilon))$.

Crucial: N does not depend on V!

Application to de-bordering

Setting:
$\beta : T_1 \to T_2$ morphism between tensor spaces with $\mathrm{im}(\beta_V)$ an affine cone spanning $T_2(V)$. For $f \in T_2(V)$ define
$R(f) := \min\{r \mid f = \sum_{i=1}^r \beta_V(h_i)\}$ (*rank*) and
$\underline{R}(f) := \min\{r \mid f = \lim_{\epsilon \to 0} \sum_{i=1}^r \beta_V(h_i(\epsilon))\}$ (*border rank*).

Application to de-bordering

Setting:
$\beta : T_1 \to T_2$ morphism between tensor spaces with $\mathrm{im}(\beta_V)$ an affine cone spanning $T_2(V)$. For $f \in T_2(V)$ define
$R(f) := \min\{r \mid f = \sum_{i=1}^r \beta_V(h_i)\}$ (*rank*) and
$\underline{R}(f) := \min\{r \mid f = \lim_{\epsilon \to 0} \sum_{i=1}^r \beta_V(h_i(\epsilon))\}$ (*border rank*).

Corollary
There is a function $F : \mathbb{N} \to \mathbb{N}$ such that for all V and $f \in T_2(V) : R(f) \leq F(\underline{R}(f))$.

Application to de-bordering

Setting:
$\beta : T_1 \to T_2$ morphism between tensor spaces with $\mathrm{im}(\beta_V)$ an affine cone spanning $T_2(V)$. For $f \in T_2(V)$ define
$R(f) := \min\{r \mid f = \sum_{i=1}^{r} \beta_V(h_i)\}$ (*rank*) and
$\underline{R}(f) := \min\{r \mid f = \lim_{\epsilon \to 0} \sum_{i=1}^{r} \beta_V(h_i(\epsilon))\}$ (*border rank*).

Corollary
There is a function $F : \mathbb{N} \to \mathbb{N}$ such that for all V and $f \in T_2(V) : R(f) \leq F(\underline{R}(f))$.

Proof: fix r and set $\alpha : T_1^r \to T_2, \alpha(h_1, \dots, h_r) := \sum_{i=1}^{r} \beta(h_i)$.
Theorem $\rightsquigarrow$ if $\underline{R}(f) \leq r$, then $\exists h_i(\epsilon) \in T_1(V) \otimes \langle \epsilon^{-N}, \dots, \epsilon^{N} \rangle$ with $\lim_{\epsilon \to 0} \sum_{i=1}^{r} \beta(h_i(\epsilon)) = f$. But then f is in the span of $\beta(h_i(t_j))$ for $i = 1, \dots, r$ and $(2\deg(\beta)N + 1)$ distinct $t_j \neq 0$
$\rightsquigarrow R(f) \leq (2\deg(\beta)N + 1)r$. $\square$

Some remarks

Main theorem [BDES, 2023]

$\alpha : X \to Y$ a morphism, then there is an $N \in \mathbb{N}$ such that for all V and all $y \in \overline{\mathrm{im}(\alpha_V)}$ there is a formal curve $x(\epsilon) \in X(V)(\mathbb{C}((\epsilon)))$ with exponents $\geq -N$ such that $y = \lim_{\epsilon \to 0} \alpha_V(x(\epsilon))$.

- *Easy* if $X(V) = V^m$: then for all V and $y \in \overline{\mathrm{im}(\alpha_V)}$ we have $y \in \overline{\mathrm{im}_{\alpha_U} : X(U) \to Y(U)}$ for some m-dimensional $U \subseteq V$. No new results for ordinary tensor rank.

Some remarks

Main theorem [BDES, 2023]

$\alpha : X \to Y$ a morphism, then there is an $N \in \mathbb{N}$ such that for all V and all $y \in \overline{\mathrm{im}(\alpha_V)}$ there is a formal curve $x(\epsilon) \in X(V)(\mathbb{C}((\epsilon)))$ with exponents $\geq -N$ such that $y = \lim_{\epsilon \to 0} \alpha_V(x(\epsilon))$.

• *Easy* if $X(V) = V^m$: then for all V and $y \in \overline{\mathrm{im}(\alpha_V)}$ we have $y \in \overline{\mathrm{im}_{\alpha_U} : X(U) \to Y(U)}$ for some m-dimensional $U \subseteq V$. No new results for ordinary tensor rank.

• But for $\beta : (S^2)^2 \to S^4, (f, g) \mapsto f \cdot g$ (quadric rank of quartics), I know no other debordering result, and no other way to prove it than using the theorem above.

Some remarks

Main theorem [BDES, 2023]

$\alpha : X \to Y$ a morphism, then there is an $N \in \mathbb{N}$ such that for all V and all $y \in \overline{\text{im}(\alpha_V)}$ there is a formal curve $x(\epsilon) \in X(V)(\mathbb{C}((\epsilon)))$ with exponents $\geq -N$ such that $y = \lim_{\epsilon \to 0} \alpha_V(x(\epsilon))$.

- *Easy* if $X(V) = V^m$: then for all V and $y \in \overline{\text{im}(\alpha_V)}$ we have $y \in \overline{\text{im}_{\alpha_U} : X(U) \to Y(U)}$ for some m-dimensional $U \subseteq V$. No new results for ordinary tensor rank.

- But for $\beta : (S^2)^2 \to S^4, (f, g) \mapsto f \cdot g$ (quadric rank of quartics), I know no other debordering result, and no other way to prove it than using the theorem above.

- For *partition rank*, $\exists$ much better bounds (Lampert!)

Ingredients of the proof

- X tensor variety $\rightsquigarrow$ sequence

... $\leftarrow X(\mathbb{C}^n) \leftarrow X(\mathbb{C}^{n+1}) \leftarrow$ ...; set $X_\infty := \lim_{\leftarrow n} X(\mathbb{C}^n)$.

X_∞ is a GL-*variety*: an infinite-dimensional variety with an action of $\mathrm{GL} = \bigcup_n \mathrm{GL}_n$; e.g. if $X(V) = V \otimes V$, then X_∞ is the space of $\mathbb{N} \times \mathbb{N}$-matrices with action $(g, A) \mapsto gAg^T$.

Ingredients of the proof

- X tensor variety $\rightsquigarrow$ sequence

... $\leftarrow X(\mathbb{C}^n) \leftarrow X(\mathbb{C}^{n+1}) \leftarrow$ ...; set $X_\infty := \lim_{\leftarrow n} X(\mathbb{C}^n)$.

X_∞ is a GL-*variety*: an infinite-dimensional variety with an action of $\mathrm{GL} = \bigcup_n \mathrm{GL}_n$; e.g. if $X(V) = V \otimes V$, then X_∞ is the space of $\mathbb{N} \times \mathbb{N}$-matrices with action $(g, A) \mapsto gAg^T$.

- $\alpha : X \to Y \rightsquigarrow$ morphism $\alpha_\infty : X_\infty \to Y_\infty$ of GL-varieties.

Ingredients of the proof

• X tensor variety $\rightsquigarrow$ sequence
... $\leftarrow X(\mathbb{C}^n) \leftarrow X(\mathbb{C}^{n+1}) \leftarrow$ ...; set $X_\infty := \lim_{\leftarrow n} X(\mathbb{C}^n)$.

X_∞ is a GL-*variety*: an infinite-dimensional variety with an action of $\mathrm{GL} = \bigcup_n \mathrm{GL}_n$; e.g. if $X(V) = V \otimes V$, then X_∞ is the space of $\mathbb{N} \times \mathbb{N}$-matrices with action $(g, A) \mapsto gAg^T$.

• $\alpha : X \to Y \rightsquigarrow$ morphism $\alpha_\infty : X_\infty \to Y_\infty$ of GL-varieties.

• $\mathrm{im}(\alpha_\infty)$ is GL-constructible (Chevalley for GL-varieties); in particular, $\mathrm{im}(\alpha_\infty) \supseteq U$: GL-stable, open, dense in $\overline{\mathrm{im}(\alpha_\infty)}$.

Ingredients of the proof

• X tensor variety $\rightsquigarrow$ sequence
$\ldots \leftarrow X(\mathbb{C}^n) \leftarrow X(\mathbb{C}^{n+1}) \leftarrow \ldots$; set $X_\infty := \lim_{\leftarrow n} X(\mathbb{C}^n)$.

X_∞ is a GL-*variety*: an infinite-dimensional variety with an action of $\mathrm{GL} = \bigcup_n \mathrm{GL}_n$; e.g. if $X(V) = V \otimes V$, then X_∞ is the space of $\mathbb{N} \times \mathbb{N}$-matrices with action $(g, A) \mapsto gAg^T$.

• $\alpha : X \to Y \rightsquigarrow$ morphism $\alpha_\infty : X_\infty \to Y_\infty$ of GL-varieties.

• $\mathrm{im}(\alpha_\infty)$ is GL-constructible (Chevalley for GL-varieties); in particular, $\mathrm{im}(\alpha_\infty) \supseteq U$: GL-stable, open, dense in $\overline{\mathrm{im}(\alpha_\infty)}$.

Curve selection theorem: Y an irreducible GL-variety, $U \subseteq Y$ open dense, $y \in Y$, then $\exists$ irreducible affine curve C and morphism $i : C \to Y$ with $y \in \mathrm{im}(i)$ and $\mathrm{im}(i) \cap U \neq \emptyset$.

(implies the Main Theorem)

Ingredients of the proof, continued

(Note: $V(x_2^2 - x_1, x_3^3 - x_1, ...) \subseteq \mathbb{C}^{\mathbb{N}}$ admits no nonconstant maps from (finite-type) curves.)

Curve selection theorem: Y an irreducible GL-variety, $U \subseteq Y$ open, $y \in Y$, then $\exists$ irreducible affine curve C and morphism $i : C \to Y$ with $y \in \mathrm{im}(i)$ and $\mathrm{im}(i) \cap U \neq \emptyset$.

Ingredients of the proof, continued

(Note: $V(x_2^2 - x_1, x_3^3 - x_1, ...) \subseteq \mathbb{C}^{\mathbb{N}}$ admits no nonconstant maps from (finite-type) curves.)

Curve selection theorem: Y an irreducible GL-variety, $U \subseteq Y$ open, $y \in Y$, then $\exists$ irreducible affine curve C and morphism $i : C \to Y$ with $y \in \text{im}(i)$ and $\text{im}(i) \cap U \neq \emptyset$.

• Unirationality: $\exists$ dominant $\beta_\infty : B \times T_\infty \to Y$, where T_∞ is an affine space and B is irreducible, finite-dimensional.

Ingredients of the proof, continued

(Note: $V(x_2^2 - x_1, x_3^3 - x_1, ...) \subseteq \mathbb{C}^{\mathbb{N}}$ admits no nonconstant maps from (finite-type) curves.)

Curve selection theorem: Y an irreducible GL-variety, $U \subseteq Y$ open, $y \in Y$, then $\exists$ irreducible affine curve C and morphism $i : C \to Y$ with $y \in \mathrm{im}(i)$ and $\mathrm{im}(i) \cap U \neq \emptyset$.

• Unirationality: $\exists$ dominant $\beta_\infty : B \times T_\infty \to Y$, where T_∞ is an affine space and B is irreducible, finite-dimensional.

• $\beta_\infty(b_0, t_0) \in U$ for some (b_0, t_0). If also $\beta_\infty(b_1, t_1) = y$, then find a curve $j : C \to B$ with $j(c_0) = b_0$ and $j(c_1) = b_1$ and $h : C \to \mathbb{C}$ with $h(c_i) = i$. Then $i(c) := \beta_\infty(j(c), (1 - h(c))t_0 + h(c)t_1))$ works.

Ingredients of the proof, conclusion

Unirationality theorem for pairs

$Z \subseteq Y$ irreducible, closed GL-subvariety, then $\exists$ diagram of GL-varieties

$$\begin{array}{ccc} A \times T_\infty & \xrightarrow{\subseteq \times \mathrm{id}} & B \times T_\infty \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\subseteq} & Y \end{array}$$

where T_∞ is a space of infinite tensors, B is an irreducible finite-dimensional affine variety with trivial GL-action, A is an irreducible closed subvariety of codimension 1 in B, and the vertical GL-equivariant morphisms are dominant.

- so if we take $Z := \overline{\mathrm{GL} \cdot y}$, we are done.
- proof of the theorem above uses blow-up and more.

Summary

• We have established a qualitative de-bordering results for very general notions of rank of tensors.

Summary

- We have established a qualitative de-bordering results for very general notions of rank of tensors.

- To prove these, we have developed a rather powerful theory of infinite-dimensional GL-varieties.

Summary

• We have established a qualitative de-bordering results for very general notions of rank of tensors.

• To prove these, we have developed a rather powerful theory of infinite-dimensional GL-varieties.

• In principle, all is algorithmic: there exists an algorithm that takes as input $\beta : T_1 \to T_2$ and r and computes $F(r)$ (Blatter-D-Ventura); however, it is very far from implemented.

Summary

• We have established a qualitative de-bordering results for very general notions of rank of tensors.

• To prove these, we have developed a rather powerful theory of infinite-dimensional GL-varieties.

• In principle, all is algorithmic: there exists an algorithm that takes as input $\beta : T_1 \to T_2$ and r and computes $F(r)$ (Blatter-D-Ventura); however, it is very far from implemented.

• Unfortunately, the proof does not give any bounds on F. However, tensor variety techniques *can* be used to get polynomial bounds for partition rank (Bik-D-Lampert-Ziegler).

Summary

• We have established a qualitative de-bordering results for very general notions of rank of tensors.

• To prove these, we have developed a rather powerful theory of infinite-dimensional GL-varieties.

• In principle, all is algorithmic: there exists an algorithm that takes as input $\beta : T_1 \to T_2$ and r and computes $F(r)$ (Blatter-D-Ventura); however, it is very far from implemented.

• Unfortunately, the proof does not give any bounds on F. However, tensor variety techniques *can* be used to get polynomial bounds for partition rank (Bik-D-Lampert-Ziegler).

Thank you!