Quantum Information Theory, Spring 2020

Practice problem set #13

You do **not** have to hand in these exercises, they are for your practice only.

1. **Symmetric subspace:**
   (a) Write out Π_2 and Π_3.
   (b) In class we wrote down a basis for $\mathrm{Sym}^2(\mathbb{C}^2)$. Write down bases of $\mathrm{Sym}^2(\mathbb{C}^d)$ and $\mathrm{Sym}^3(\mathbb{C}^2)$.
   (c) Verify that $R_\pi R_\tau = R_{\pi\tau}$ and $R_\pi^\dagger = R_{\pi^{-1}}$, for all $\pi, \tau \in S_n$.
   (d) Verify that $\Pi_n = \frac{1}{n!}\sum_{\pi \in S_n} R_\pi$ is the orthogonal projection onto the symmetric subspace.

2. **Integral formula:** In this exercise you can prove the integral formula:

$$\Pi_n = \binom{n+d-1}{n} \int |\psi\rangle^{\otimes n}\langle\psi|^{\otimes n} d\psi =: \tilde{\Pi}_n$$

   (a) Show that $\tilde{\Pi}_n = \Pi_n \tilde{\Pi}_n$.
   (b) Recall the following important fact from Lemma 13.10: *If* $A \in L(\mathcal{H}^{\otimes n})$ *is an operator such that* $[A, U^{\otimes n}] = 0$ *for all unitaries* $U \in U(\mathcal{H})$, *then* A *is a linear combination of permutation operators* R_π, $\pi \in S_n$. Use this fact to show that $\tilde{\Pi}_n = \sum_\pi c_\pi R_\pi$ for suitable $c_\pi \in \mathbb{C}$.
   (c) Use parts (a) and (b) to prove the integral formula. That is, show that $\tilde{\Pi}_n = \Pi_n$.

3. **Haar measure:** There is a unique probability measure dU on the unitary operators $U(\mathcal{H})$ that is invariant under $U \mapsto VUW$ for every pair of unitaries V, W. It is called the *Haar measure*. Its defining property can be stated as follows: For every continuous function f on $U(\mathcal{H})$ and for all unitaries $V, W \in U(\mathcal{H})$, it holds that $\int f(U)\, dU = \int f(VUW)\, dU$. Now let $A \in L(\mathcal{H})$.
   (a) Argue that $\int UAU^\dagger dU$ commutes with all unitaries.
   (b) Deduce that $\int UAU^\dagger dU = \frac{\mathrm{Tr}[A]}{d} I$, where $d = \dim \mathcal{H}$.

4. ⚗ **De Finetti theorem and quantum physics (optional):** Given a Hermitian operator h on $\mathbb{C}^d \otimes \mathbb{C}^d$, consider the operator $H = \frac{1}{n-1}\sum_{i \neq j} h_{i,j}$ on $(\mathbb{C}^d)^{\otimes n}$, where $h_{i,j}$ acts by h on subsystems i and j and by the identity on the remaining subsystems (e.g., $h_{1,2} = h \otimes I^{\otimes(n-2)}$).

   (a) Show that $\frac{E_0}{n} \leqslant \frac{1}{n}\langle\psi^{\otimes n}|H|\psi^{\otimes n}\rangle = \langle\psi^{\otimes 2}|h|\psi^{\otimes 2}\rangle$ for every pure state ψ on $\mathbb{C}^d$.

   Let E_0 denote the smallest eigenvalue of H and $|E_0\rangle$ a corresponding eigenvector. If the eigenspace is one-dimensional and $n > d$ then $|E_0\rangle \in \mathrm{Sym}^n(\mathbb{C}^d)$ (you do not need to show this).

   (b) Use the de Finetti theorem to show that $\frac{E_0}{n} \approx \min_{\|\psi\|=1}\langle\psi^{\otimes 2}|h|\psi^{\otimes 2}\rangle$ for large n.

   Interpretation: The Hamiltonian H *describes a mean-field system. Your result shows that in the thermodynamic limit the ground state energy density can be computed using states of form* $\psi^{\otimes n}$.