Quantum Information Theory, Spring 2020

Homework problem set #13 **due May 27, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(4 points)* **Rényi-2 entropy:** In this problem you will study a new entropy measure called the *Rényi-2 entropy*. It is defined by $H_2(\rho) := -\log \mathrm{Tr}[\rho^2]$ for any quantum state $\rho \in D(\mathbb{C}^d)$.

   (a) Find a formula for $H_2(\rho)$ in terms of the eigenvalues of ρ.
   (b) Show that $H_2(\rho) \leqslant H(\rho)$ by using Jensen's inequality.
   (c) Show that $\mathrm{Tr}[\rho^2] = \mathrm{Tr}[F\rho^{\otimes 2}]$, where $F : |i\rangle \otimes |j\rangle \mapsto |j\rangle \otimes |i\rangle$ for all $i, j \in \{1, \ldots, d\}$, is the *swap operator*.

2. *(4 points)* **Average entanglement:** In this exercise you will study the average entanglement of a random pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$ drawn from the uniform distribution $d\psi_{AB}$ discussed in class. Recall that the entanglement entropy of a pure state $|\psi_{AB}\rangle$ is given by $H(\rho_A) = H(\rho_B)$, where ρ_A and ρ_B are the reduced states of $|\psi_{AB}\rangle$.

   (a) Let F_{AA}, F_{BB} denote the swap operators on $\mathcal{H}_A^{\otimes 2}, \mathcal{H}_B^{\otimes 2}$ and let $d_A = \dim \mathcal{H}_A$, $d_B = \dim \mathcal{H}_B$. Use the integral formula for the symmetric subspace to deduce that

   $$\int |\psi_{AB}\rangle^{\otimes 2} \langle \psi_{AB}|^{\otimes 2} \, d\psi_{AB} = \frac{1}{d_A d_B (d_A d_B + 1)} \left(I_{AA} \otimes I_{BB} + F_{AA} \otimes F_{BB} \right).$$

   (b) Verify that $\int \mathrm{Tr}[\rho_A^2] \, d\psi_{AB} = \frac{d_A + d_B}{d_A d_B + 1}$.
   (c) Show that the average Rényi-2 entropy $H_2(\rho_A)$ for a random pure state $|\psi_{AB}\rangle$ is at least $\log(\min(d_A, d_B)) - 1$. Conclude that the same holds for the entanglement entropy.

   Hint: Use Problem 1 and Jensen's inequality.

3. *(4 points)* **Haar measure:** In the exercise class, we discussed the Haar measure on $U(\mathcal{H})$, which is the unique probability measure dU with the following property: For every continuous function f on $U(\mathcal{H})$ and for all unitaries $V, W \in U(\mathcal{H})$, it holds that $\int f(U) \, dU = \int f(VUW) \, dU$.

   (a) Argue that, for any operator $A \in L(\mathcal{H}^{\otimes n})$, the so-called *twirl* $\int U^{\otimes n} A U^{\dagger \otimes n} \, dU$ can always be written as a linear combination of permutation operators R_π, $\pi \in S_n$.
   (b) Deduce that $\int U^{\otimes 2} A U^{\dagger \otimes 2} \, dU = \alpha I + \beta F$ for every $A \in L(\mathcal{H}^{\otimes 2})$, where F is the swap operator on $\mathcal{H}^{\otimes 2}$, $\alpha = \frac{d}{d^3 - d} \mathrm{Tr}[A] - \frac{1}{d^3 - d} \mathrm{Tr}[FA]$, and $\beta = \frac{d}{d^3 - d} \mathrm{Tr}[FA] - \frac{1}{d^3 - d} \mathrm{Tr}[A]$.