Quantum Information Theory, Spring 2020

Homework problem set #12 **due May 18, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(4 points)* **Fidelity inequality:** Let $\mathcal{H}$ be a Hilbert space with $\dim(\mathcal{H}) \geqslant 2$.

   (a) Let $|u_1\rangle, |u_2\rangle, |v\rangle \in \mathcal{H}$ be arbitrary pure quantum states. Show that

   $$|\langle u_1|v\rangle|^2 + |\langle u_2|v\rangle|^2 \leqslant 1 + |\langle u_1|u_2\rangle|.$$

   Hint: Upper bound the left-hand side by the largest eigenvalue of some rank-2 matrix. Compute this eigenvalue to get the right-hand side.

   (b) Let $\rho_1, \rho_2, \sigma \in D(\mathcal{H})$ be arbitrary states. Show that

   $$F(\rho_1, \sigma)^2 + F(\rho_2, \sigma)^2 \leqslant 1 + F(\rho_1, \rho_2).$$

2. *(4 points)* **Entanglement cost using compression and teleportation:**
   In this exercise you will give an alternative proof for the fact that the entanglement cost is at most the entanglement entropy for a pure state. Let $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a pure state.

   (a) Let ρ_A and ρ_B be its reduced density matrices and let $\alpha > H(\rho_A) = H(\rho_B)$. Show, using compression, that for all $\delta > 0$ and all but finitely many n there exists an LOCC protocol which converts $\phi^{\otimes \lfloor \alpha n \rfloor}$ into a state $\tilde{\rho}_n$ with $F(\rho_{AB}^{\otimes n}, \tilde{\rho}_n) > 1 - \delta$.
   Hint: Use teleportation!

   (b) Use (a) to show that $E_C(\rho_{AB}) \leqslant H(\rho_A)$, for every pure state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$.

3. *(4 points)* **Entanglement rank and the fidelity with the maximally entangled state:**
   Let $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^{\otimes n}$ and $\phi_n \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ be the canonical maximally entangled state of dimension n, i.e., $\phi_n = |\Phi_n^+\rangle\langle\Phi_n^+|$ where

   $$|\Phi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle \otimes |i\rangle.$$

   Show that $F(\sigma, \phi_n)^2 \leqslant r/n$, for any (mixed) state $\sigma \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ of entanglement rank r.