Quantum Information Theory, Spring 2020

Homework problem set #11 **due May 11, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(4 points)* **Entanglement entropy and separable maps:** Let $|\Psi_1\rangle_{AB}$ be a pure state on registers A and B, and assume that it can be perfectly transformed to another state $|\Psi_2\rangle_{AB}$ by a separable operation. Show that such transformation cannot make the state more entangled in the sense of increasing its entanglement entropy. That is, show that

$$H(\rho_1) \geqslant H(\rho_2),$$

where $\rho_i = \mathrm{Tr}_B\left[|\Psi_i\rangle\langle\Psi_i|_{AB}\right]$ denotes the reduced state of $|\Psi_i\rangle_{AB}$ on Alice and $H(\rho)$ denotes the von Neumann entropy of ρ. *Hint: Entropy is a concave function.*

2. *(4 points)* **Local conversion with *no* communication:** Show that a pure state $|\Psi_1\rangle_{AB}$ shared by Alice and Bob can be converted to another pure state $|\Psi_2\rangle_{AB}$ using *only* local unitary operations (and *no* communication) if and only if

$$\rho_1 \prec \rho_2 \qquad \text{and} \qquad \rho_2 \prec \rho_1$$

where $\rho_i = \mathrm{Tr}_B\left[|\Psi_i\rangle\langle\Psi_i|_{AB}\right]$ denotes the reduced state of $|\Psi_i\rangle_{AB}$ on Alice. Show both directions of the implication.

3. *(4 points)* **Nielsen's theorem in action:** According to Nielsen's theorem, a maximally entangled state $|\Psi_1\rangle_{AB}$ shared between Alice and Bob can be transformed to any other shared pure state $|\Psi_2\rangle_{AB}$ of the same local dimensions by a one-way LOCC protocol from Bob to Alice. For each case below, devise an explicit one-way LOCC protocol that transforms $|\Psi_1\rangle_{AB}$ to $|\Psi_2\rangle_{AB}$ and succeeds with 100% probability. Write down the Kraus operators of Bob's instrument and the unitary corrections that Alice must apply after she receives Bob's measurement outcome.

   (a) Let $p \in [0, 1]$ and

$$|\Psi_1\rangle_{AB} = \frac{1}{\sqrt{2}}\,|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \otimes |1\rangle,$$
$$|\Psi_2\rangle_{AB} = \sqrt{p}\,|0\rangle \otimes |0\rangle + \sqrt{1-p}\,|1\rangle \otimes |1\rangle.$$

   (b) Let $p \in P(\mathbb{Z}_d)$ be an arbitrary probability distribution over $\mathbb{Z}_d = \{0, \ldots, d-1\}$ and

$$|\Psi_1\rangle_{AB} = \sum_{i \in \mathbb{Z}_d} \frac{1}{\sqrt{d}}\,|i\rangle \otimes |i\rangle,$$
$$|\Psi_2\rangle_{AB} = \sum_{i \in \mathbb{Z}_d} \sqrt{p(i)}\,|i\rangle \otimes |i\rangle.$$

   Hint: Let $S : \mathbb{C}^{\mathbb{Z}_d} \to \mathbb{C}^{\mathbb{Z}_d}$ denote the cyclic shift operator that acts as $S|i\rangle = |i+1\rangle$ where "+" denotes addition modulo d. Notice that $\frac{1}{d}\sum_{a \in \mathbb{Z}_d} S^a p = u$ where p is the original probability distribution and $u = (1, \ldots, 1)/d$ is the uniform distribution on $\mathbb{Z}_d$.

4. *(2 bonus points)* **Practice:** Implement a subroutine that, given two probability distributions p and q (not necessarily of the same length) determines whether $p \prec q$.

   (a) The file `abc.txt` contains three probability distributions: a, b, and c. Compare the distributions a and b using your subroutine and output `"a < b"`, `"b < a"`, or `"incomparable"`.

   (b) Use your subroutine to compare the distributions $a \otimes c$ and $b \otimes c$.
   Output `"a*c < b*c"`, `"b*c < a*c"`, or `"incomparable"`.

   (c) How can you interpret this outcome?

   (d) The files `psi1.txt` and `psi2.txt` contain bipartite pure states

   $$|\Psi_1\rangle_{AB} \in \mathbb{C}^5 \otimes \mathbb{C}^7 \qquad \text{and} \qquad |\Psi_2\rangle_{AB} \in \mathbb{C}^5 \otimes \mathbb{C}^9,$$

   where Alice's dimension is 5 and Bob's dimensions are 7 and 9, respectively. Output the eigenvalues of the reduced states on Alice's system A and determine whether $|\Psi_1\rangle_{AB}$ can be perfectly transformed into $|\Psi_2\rangle_{AB}$ by LOCC.