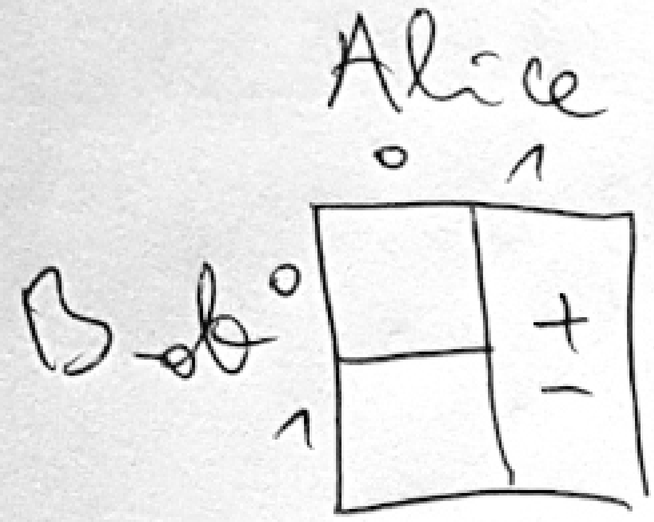
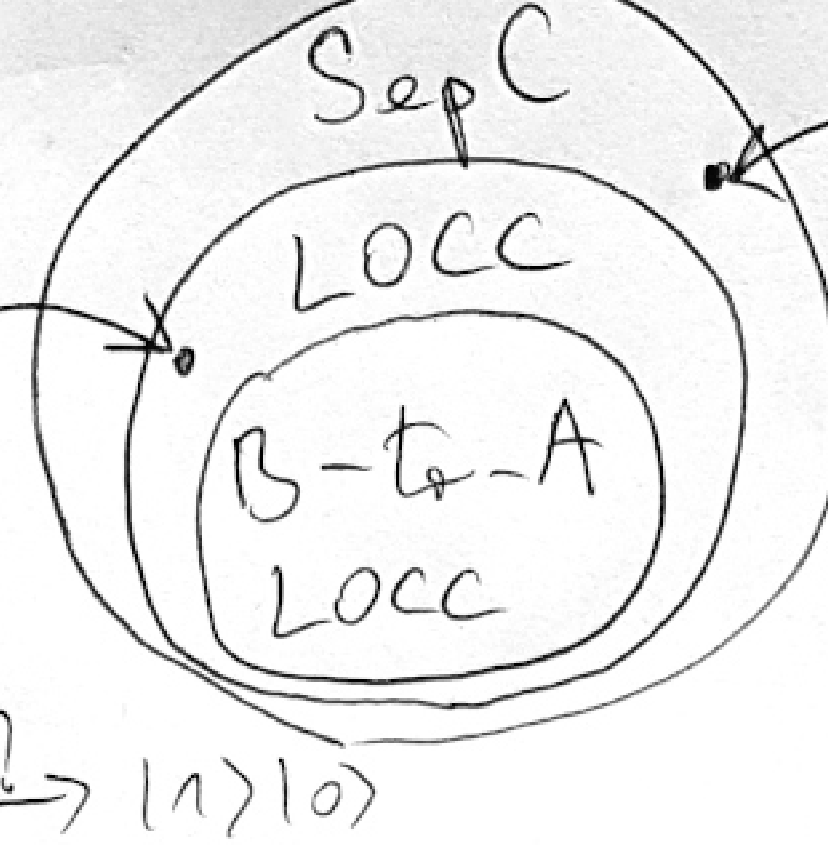
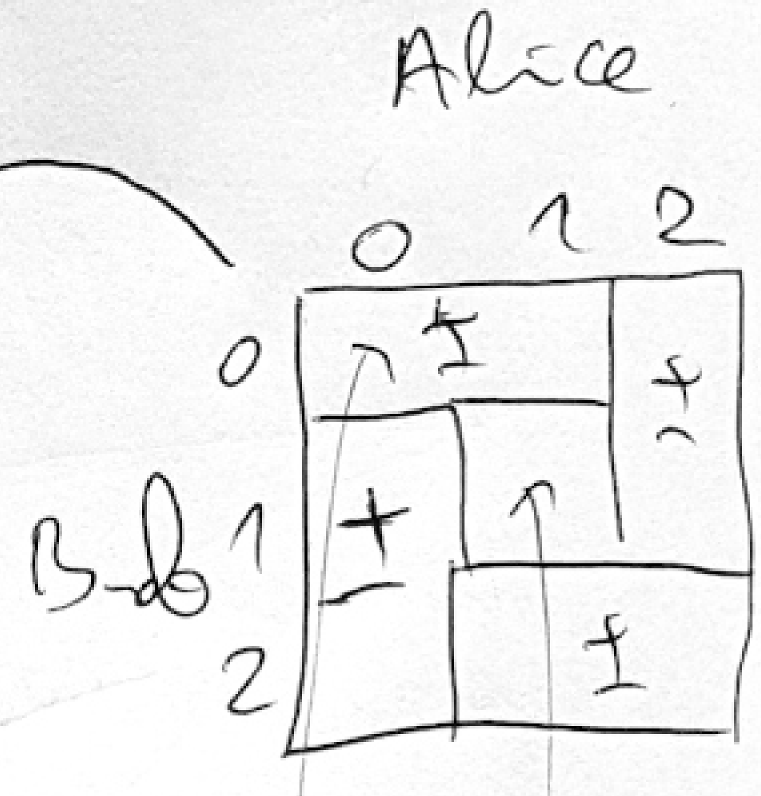
Lecture 11







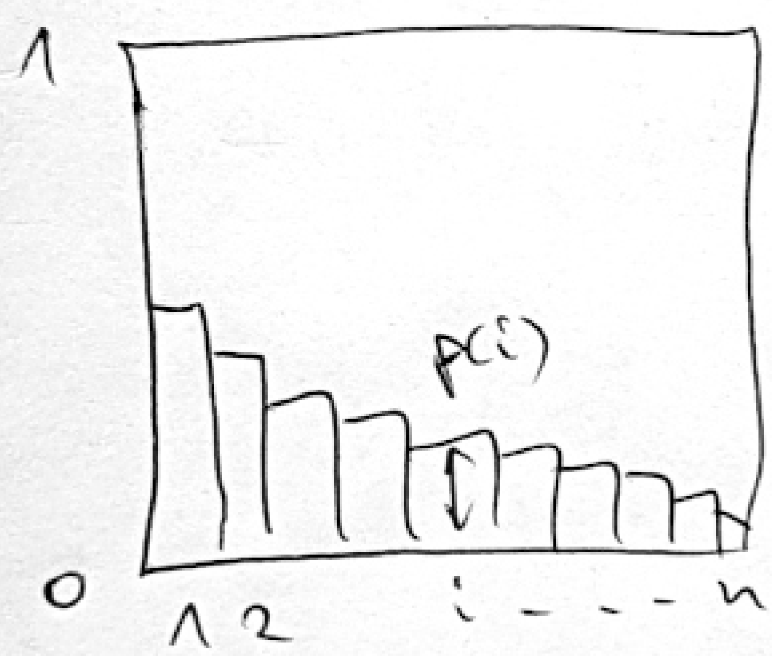
$|0\rangle|0\rangle$ $\quad$ $|1\rangle|+\rangle$
$|0\rangle|1\rangle$ $\quad$ $|1\rangle|-\rangle$ $\Big\}\overset{?}{\longmapsto}$ $|1\rangle|0\rangle$
$|1\rangle|1\rangle$

$|1\rangle|1\rangle$

$|\pm\rangle|0\rangle$

State conversion by LOCC
(specifically, $|u_{AB}\rangle \mapsto |v_{AB}\rangle$).

Majorization and wealth inequality



$p(i) \geq 0$ – wealth of person i

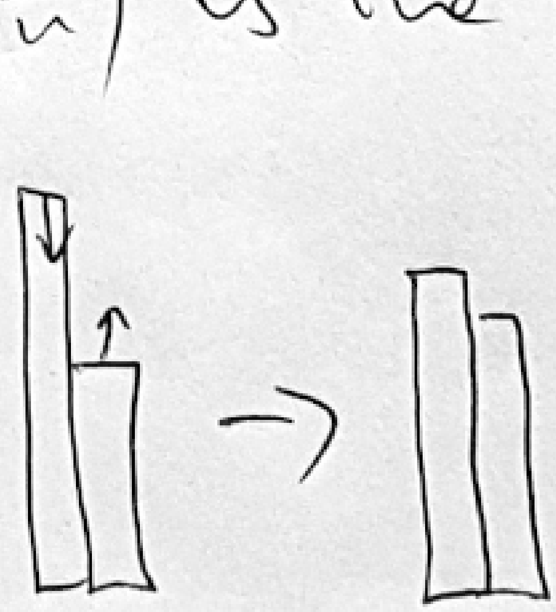
$\sum_{i=1}^{n} p(i) = 1$

How unequal is the distribution p?

Clearly, $p = (1, 0, \dots, 0)$ is the most unequal.

Also, $q = (\frac{1}{n}, \dots, \frac{1}{n})$ is the most equal.

Robin Hood move:

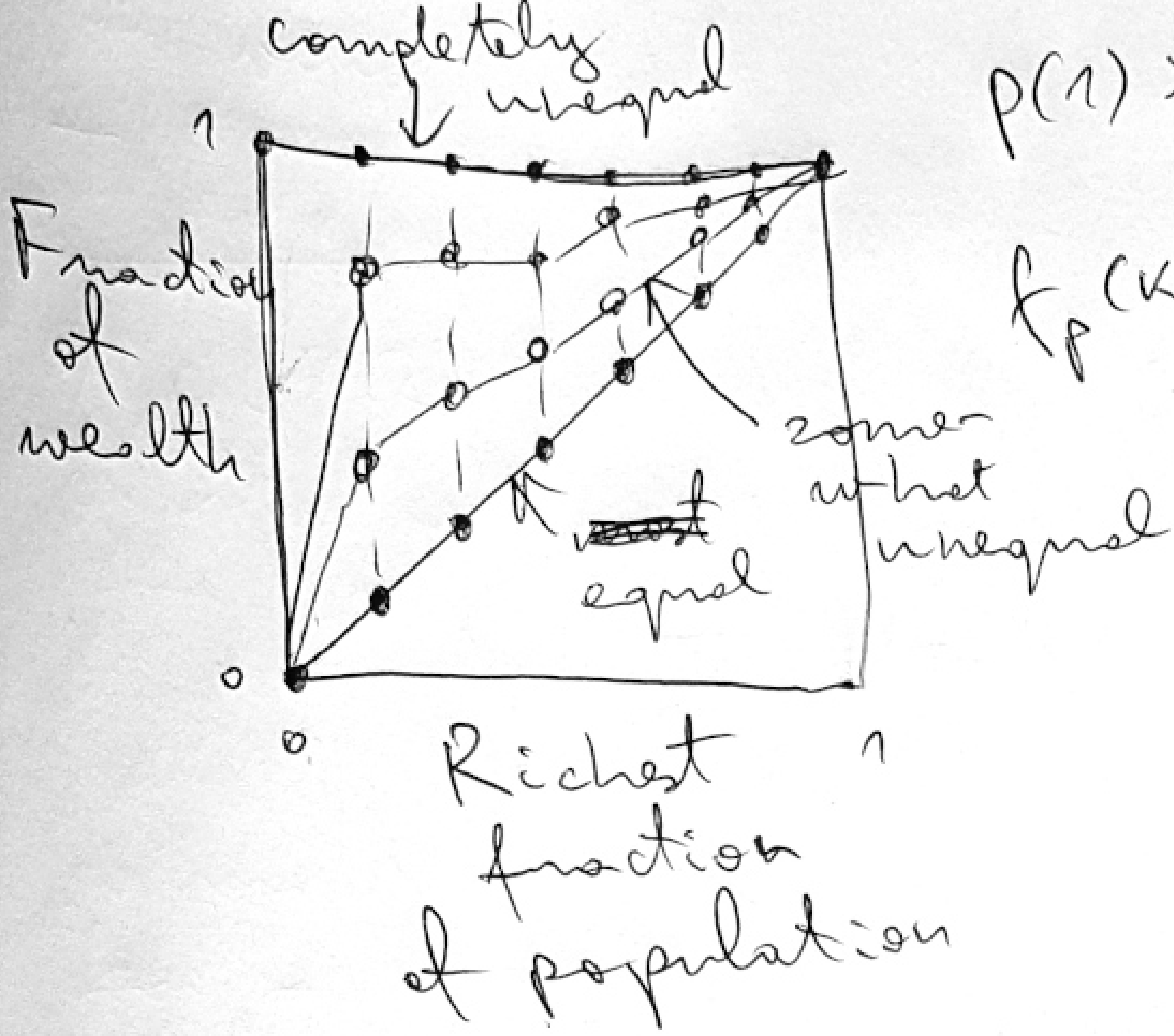


$M(c) = c\,I + (1-c)X$

$c \in [0,1]$

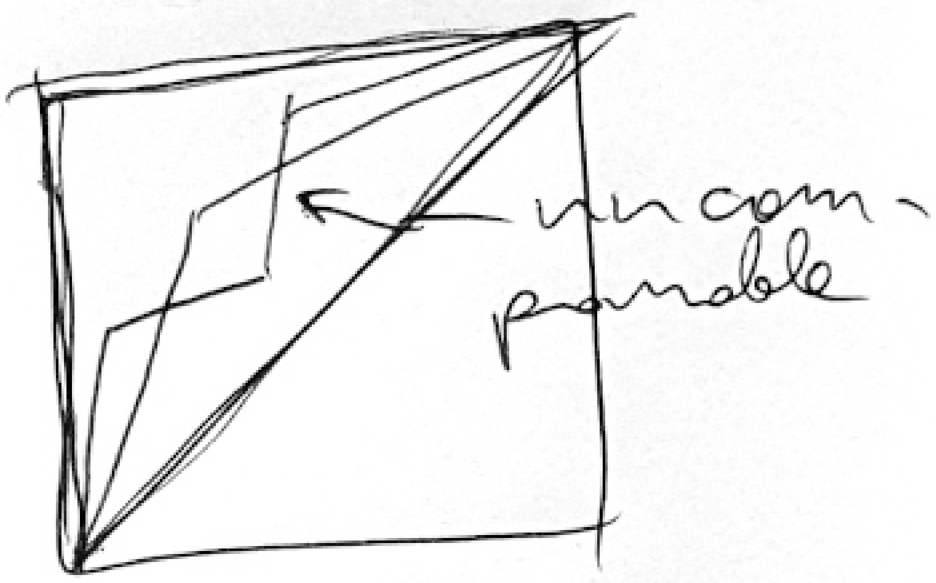
$= \begin{pmatrix} c & 1-c \\ 1-c & c \end{pmatrix}$

If $c \in (0,1)$, this always moves the distr. more equal.

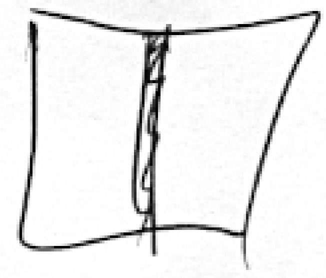


$$p(1) \geq p(2) \geq \dots \geq p(n)$$

$$f_p(k) = \sum_{i=1}^{k} p(i)$$

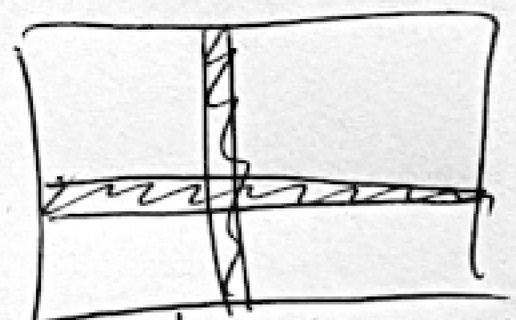


Def $A \in L(\mathbb{R}^\Sigma)$



- A is <u>stochastic</u> if
  1. $A_{ij} \geq 0 \quad \forall i,j \in \Sigma$
  2. $\sum_{i \in \Sigma} A_{ij} = 1 \quad \forall j \in \Sigma$

  } each column of A is a prob. distrib.
- A is <u>doubly stochastic</u> if A is stochastic and
  3. $\sum_{j \in \Sigma} A_{ij} = 1 \quad \forall i \in \Sigma$



- A is a <u>permutation</u> if A is doubly stochastic and 4. $A_{ij} \in \{0,1\} \quad \forall i,j \in \Sigma$

$$\begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}$$

stochastic $\Leftrightarrow$ map prob distributions to prob. distr.
(q → channel)

permutations $\Leftrightarrow$ stochastic and inverse also stochastic
(q → unitaries) $\searrow$ Sym(Σ)

Any convex combination of permutation matrices is doubly stochastic.

Converse:

<u>Theorem</u> (Birkhoff - von Neumann)

$A \in L(\mathbb{R}^{\Sigma})$ is doubly stochastic iff there exists a prob. distr. p on $\mathrm{Sym}(\Sigma)$ s.t.

$$A = \sum_{\pi \in \mathrm{Sym}(\Sigma)} p(\pi) V_\pi$$

where $V_\pi \in L(\mathbb{R}^{\Sigma})$ is the permutation matrix for π: $(V_\pi)_{ij} = \delta_{i, \pi(j)}$.

<u>Fact</u> Any doubly stochastic matrix can be obtained by a sequence of Robin Hood moves.

<u>Def</u> Let $u, v \in \mathbb{R}^{\Sigma}$. Then u <u>majorizes</u> v if $v = Au$, for some doubly stochastic A.

Notation: $u \succ v$ or $v \prec u$.

Let's sort u (reversely) by defining

$$r_1(u) \geq r_2(u) \geq \dots \geq r_n(u)$$

and $\{r_1(u), \dots, r_n(u)\} = \{u_1, \dots, u_n\}$.

<u>Theorem</u> $u, v \in \mathbb{R}^{\Sigma}$. Then $v \prec u$ iff

$$\sum_{i=1}^{m} r_i(v) \leq \sum_{i=1}^{m} r_i(u) \quad \forall m \in \{1, \dots, n-1\}$$

and

$$\sum_{i=1}^{n} r_i(v) = \sum_{i=1}^{n} r_i(u).$$

Majorization for Hermitian operators

The quantum version of permutation is a unitary matrix. And doubly stochastic matrices become

<u>Def</u> $\Phi \in C(\mathcal{H})$ is <u>mixed-unitary</u> if

$$\Phi(M) = \sum_{i \in \Sigma} p(i)\, U_i M U_i^{\dagger}$$

for some $p \in P(\Sigma)$ and $U_i \in U(\mathcal{H})$.

<u>Def</u> Let A and B be Hermitian ops on $\mathcal{H}$.
Then A majorizes B if $B = \Phi(A)$, for some mixed-unitary channel $\Phi \in C(\mathcal{H})$.
Notation: $A \succ B$ or $B \prec A$.

<u>Theorem</u> (Uhlmann) $B \prec A$ iff $\lambda(B) \prec \lambda(A)$, where $\lambda(A) \subset \mathbb{R}$ is the spectrum of A.

Alice Bob

$|u_{AB}\rangle$

$\Big\downarrow$ LOCC

$|v_{AB}\rangle$

Can they do this by LOCC?

Theorem (Nielsen)

Let $|u_{AB}\rangle, |v_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. The following are equivalent:

1. $\mathrm{Tr}_B[|u\rangle\langle u|] \prec \mathrm{Tr}_B[|v\rangle\langle v|]$
2. $\Xi[|u\rangle\langle u|] = |v\rangle\langle v|$ for some one-way LOCC protocol $\Xi \in \mathrm{LOCC}(A:B)$ from Alice to Bob.
3. Same, but Ξ is one-way from Bob to Alice.
4. Same, but $\Xi \in \mathrm{SepC}(A:B)$.

Proof $(1 \Rightarrow 2)$ Exercise: $L, R \in L(\mathcal{H}_A, \mathcal{H}_B)$

$$\mathrm{Tr}_A\left[|L\rangle\langle R|\right] = LR^\dagger \in L(\mathcal{H}_B)$$

If $L = |b\rangle\langle a|$ then $|L_{AB}\rangle = |a\rangle \otimes |b\rangle$.

st. basis vectors

Let $X, Y \in L(\mathcal{H}_A, \mathcal{H}_B)$ s.t. $|X\rangle = |u\rangle$, $|Y\rangle = |v\rangle$.

Then 1. is equivalent to

$$XX^\dagger \prec YY^\dagger$$

By ~~dk~~, there exists a mixed-unitary channel s.t.

$$XX^\dagger = \sum_{i\in\Sigma} p(i)\, W_i YY^\dagger W_i^\dagger$$

$$= \underbrace{\left(\sum_{i\in\Sigma} \sqrt{p(i)}\,(W_i Y) \otimes \langle i|\right)}_{Z} \underbrace{\left(\sum_{i\in\Sigma} \sqrt{p(i)}\,(W_i Y)^\dagger \otimes |i\rangle\right)}_{Z^\dagger} = ZZ^\dagger$$

Given $XX^\dagger = ZZ^\dagger$, how are X and Z related?

The singular value decomposition:

$$X = \sum_{k=1}^{r} s_k \, |x_k\rangle\langle y_k| \qquad r = \mathrm{rank}(X)$$

singular values

left / right singular vectors (orthonormal)

$$XX^\dagger = \sum_{j,k=1}^{r} s_j |x_j\rangle\langle y_j| \cdot s_k |y_k\rangle\langle x_k|$$

$$= \sum_{k=1}^{r} s_k^2 \, |x_k\rangle\langle x_k| = ZZ^\dagger$$

Conclude that the singular val. dec. of Z

$$Z = \sum_{k=1}^{r} s_k \, |x_k\rangle\langle w_k|$$

for some orthonormal set $\{|w_k\rangle : k = 1, \dots, r\}$.

Define $V|y_k\rangle = |w_k\rangle$, so that

$$XV^\dagger = Z = \sum_{i \in \Sigma} \sqrt{p(i)} \left(W_i Y \right) \otimes \langle i|$$

Let $|u\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle |i\rangle$

$$\mathrm{Tr}_B [|u\rangle\langle u|] = \frac{1}{n} \sum_{i=1}^{n} |i\rangle\langle i|$$

spectrum of this is $q = (\frac{1}{n}, \dots, \frac{1}{n})$.

$q \prec$ spectrum of $\mathrm{Tr}_B[|v\rangle\langle v|]$.

always true!

$\Rightarrow$ The maximally ~~mixed~~ entangled state can be converted to any other pure state by one-way LOCC.

If $p = (1, 0, \ldots, 0)$

$q = (\frac{1}{n}, \ldots, \frac{1}{n})$

then for any s, a prob. dist.,

$$q \prec s$$

In particular, $q \prec p$.